Statistics and Transport in the Restricted Three-Body Problem

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Outline of talk

- ***Some problems in dynamical astronomy suggest a three-body analysis***
  - e.g., Jupiter-family comets and scattered Kuiper Belt objects (under Neptune's control)
  - By applying dynamical systems methods to the planar, circular restricted three-body problem, several questions regarding these populations may be addressed
  - Comparison with observational data is made

Dynamical astronomy

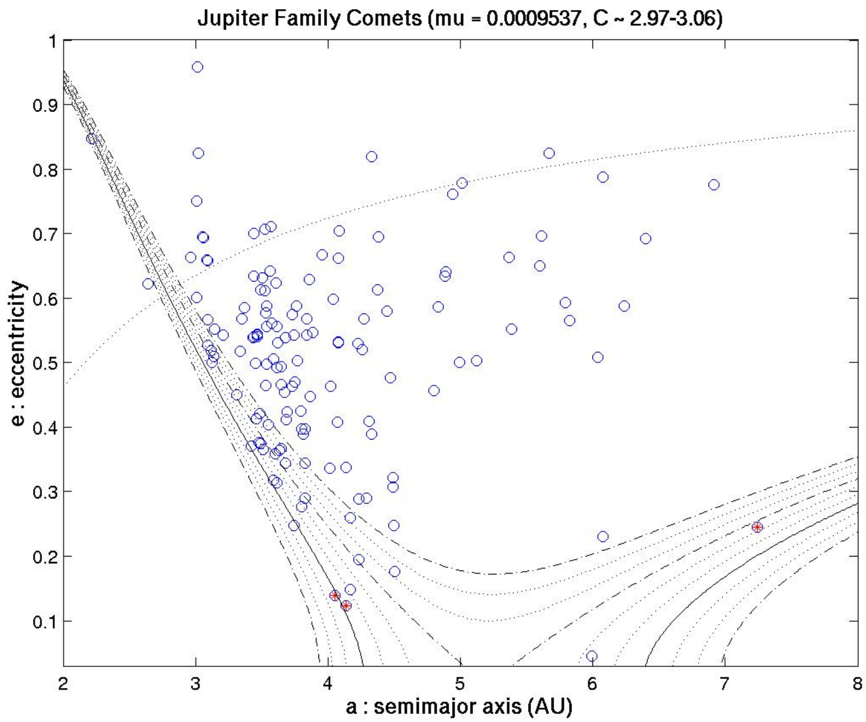
- We want to answer several questions regarding the transport and origin of some kinds of solar system material
  - How do we characterize the motion of Jupiter-family comets (JFCs) and scattered Kuiper Belt objects (SKBOs)?
  - How likely is a transition between the exterior and interior regions (e.g., Oterma)?
  - How probable is a Shoemaker-Levy 9-type collision with Jupiter? Or an asteroid collision with Earth (e.g., KT impact)?
- Harder questions
  - How does an SKBO become a JFC (and vice versa)?
  - How does impact ejecta get from Mars to Earth?

Jupiter-family comets

- JFCs and lines of constant **Tisserand parameter**,

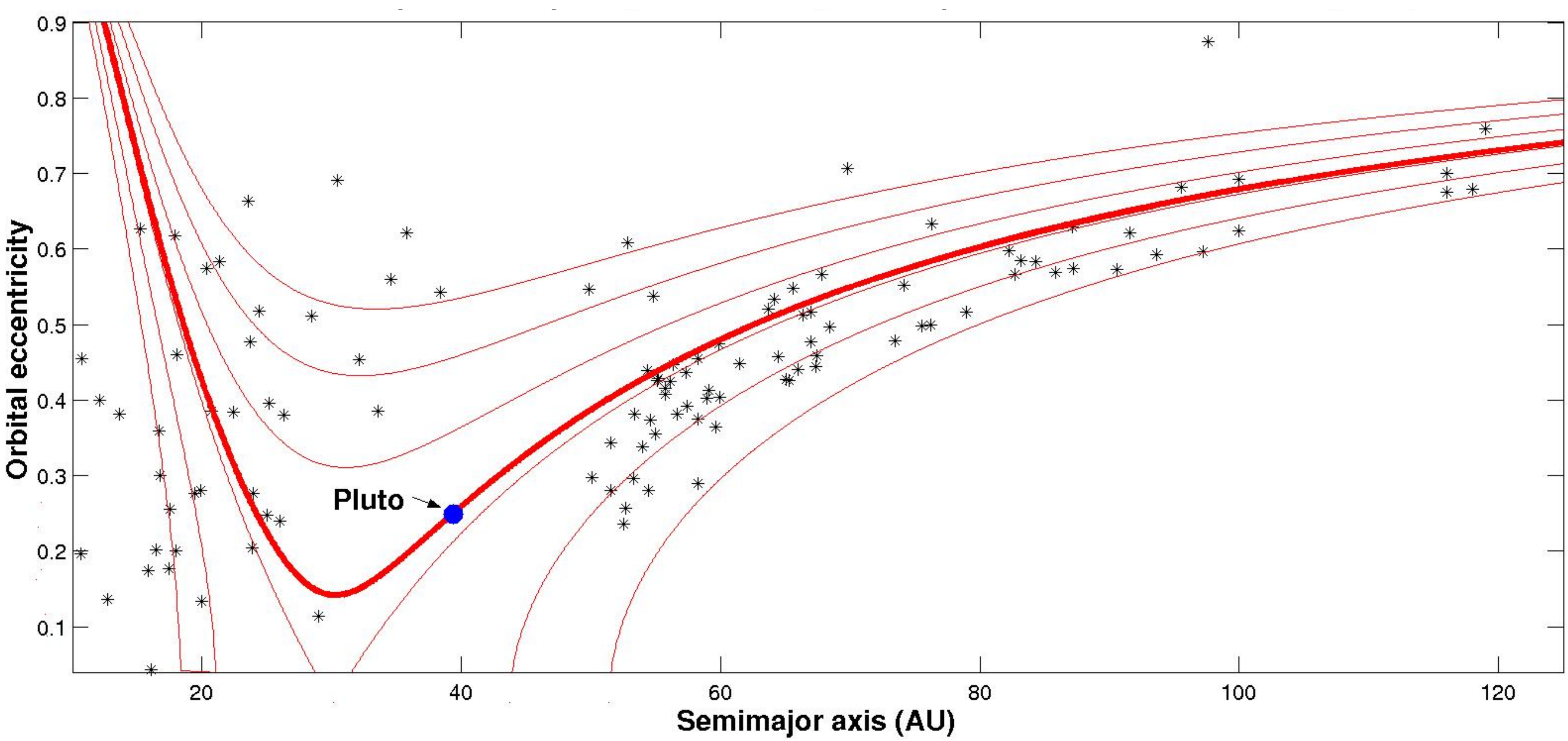
$$T = \frac{1}{a} + 2\sqrt{a(1-e^2)},$$

an approximation of the Jacobi constant



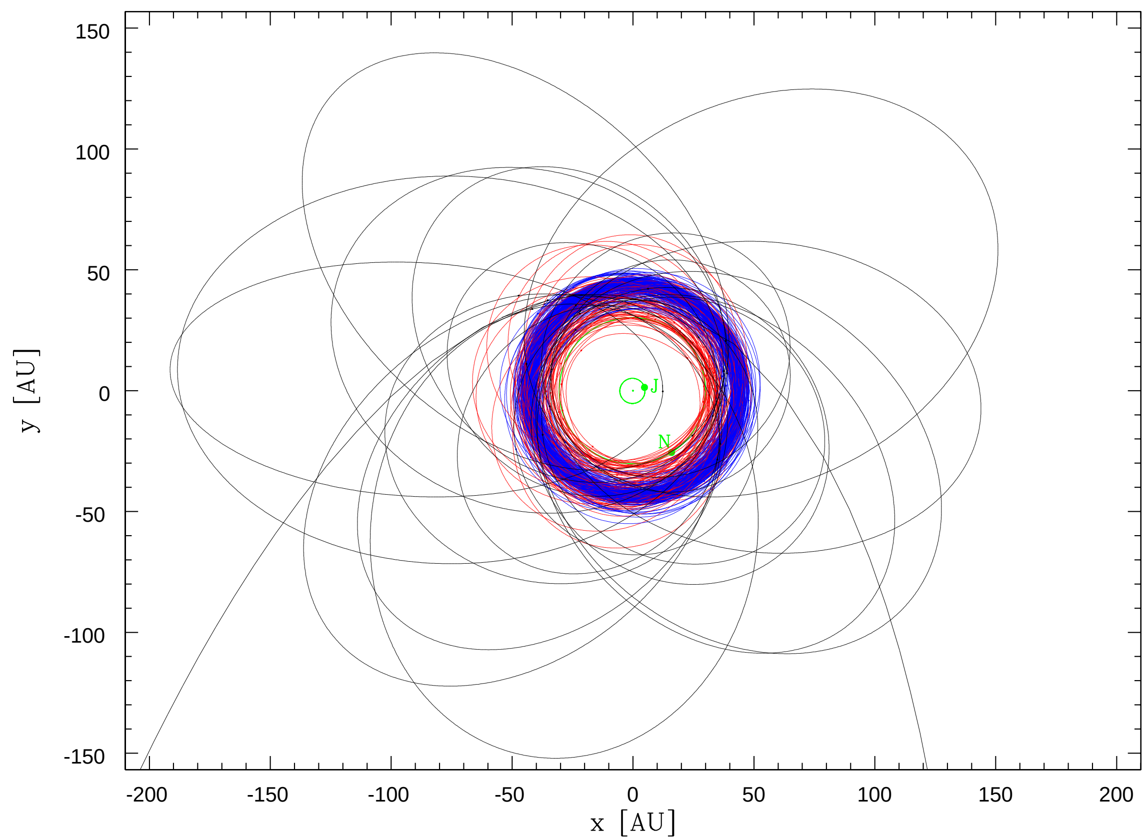
Scattered Kuiper Belt Objects

- Current SKBO locations in black, with some Tisserand values w.r.t. Neptune in red ($T \approx 3$)



Scattered Kuiper Belt Objects

- Seen in inertial space



Motion of JFCs and SKBOs

- Theory, observation, and numerical experiment show motion along nearly constant Tisserand parameter (most of the time)
- We approximate the short-timescale motion of JFCs and SKBOs as occurring within an energy shell of the restricted three-body problem
- Several objects may be in nearly the same energy shell, i.e., all have $|T - T^*| \leq \delta T$
- Can we analyze the structure of an energy shell to determine likely locations of JFCs and SKBOs?

Motion within energy shell

- Recall the planar, circular restricted three-body problem from Jerry Marsden's talk
- For fixed μ, an energy shell (or energy manifold) of energy ε is

$$\mathcal{M}(\mu, \varepsilon) = \{(x, y, \dot{x}, \dot{y}) \mid E(x, y, \dot{x}, \dot{y}) = \varepsilon\}.$$

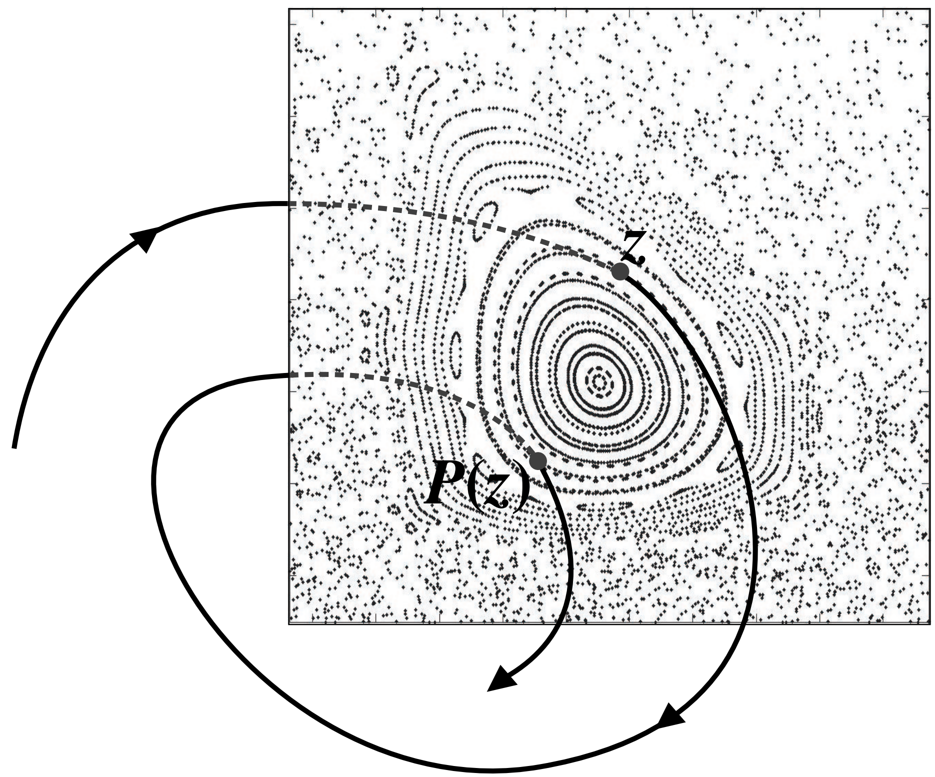
The $\mathcal{M}(\mu, \varepsilon)$ are 3-dimensional surfaces foliating the 4-dimensional phase space.

Poincaré surface-of-section

- Study Poincaré surface of section at fixed energy ε:

$$\Sigma_{(\mu,\varepsilon)} = \{(x, \dot{x}) | y = 0, \dot{y} = f(x, \dot{x}, \mu, \varepsilon) < 0\}$$

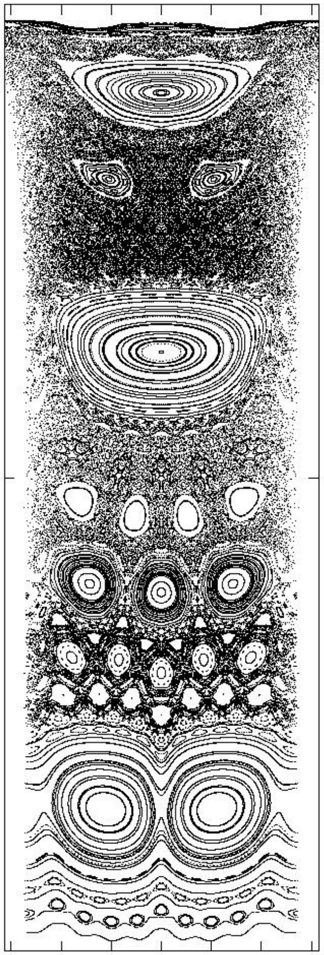
reducing the system to an area preserving map on the plane. Motion takes place on the cylinder, $S^1 \times \mathbb{R}$.



Poincaré surface-of-section and map P

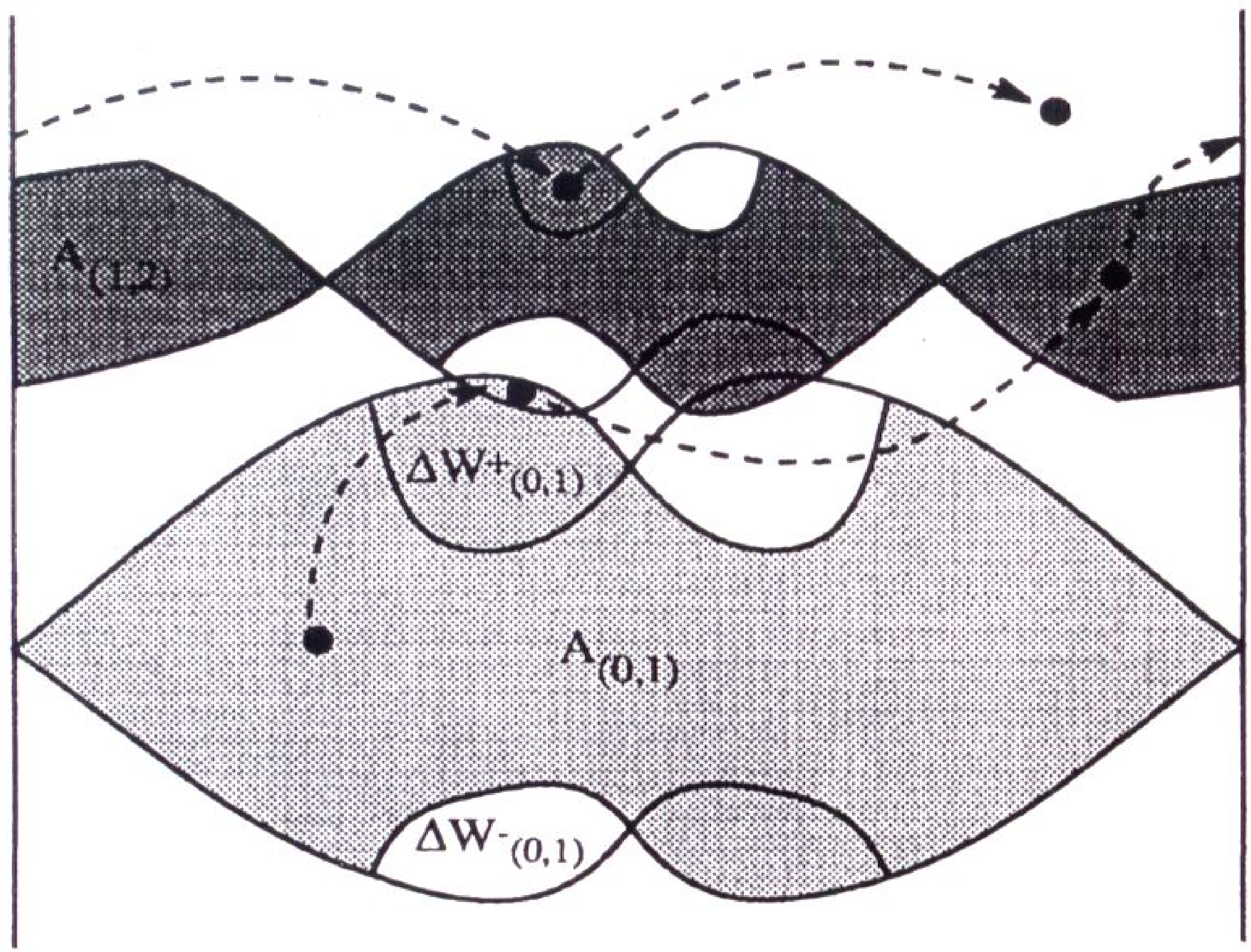
Poincaré surface-of-section

- The energy shell has regular components (KAM tori) and irregular components. Large connected irregular component is the **"chaotic sea."**



Movement among resonances

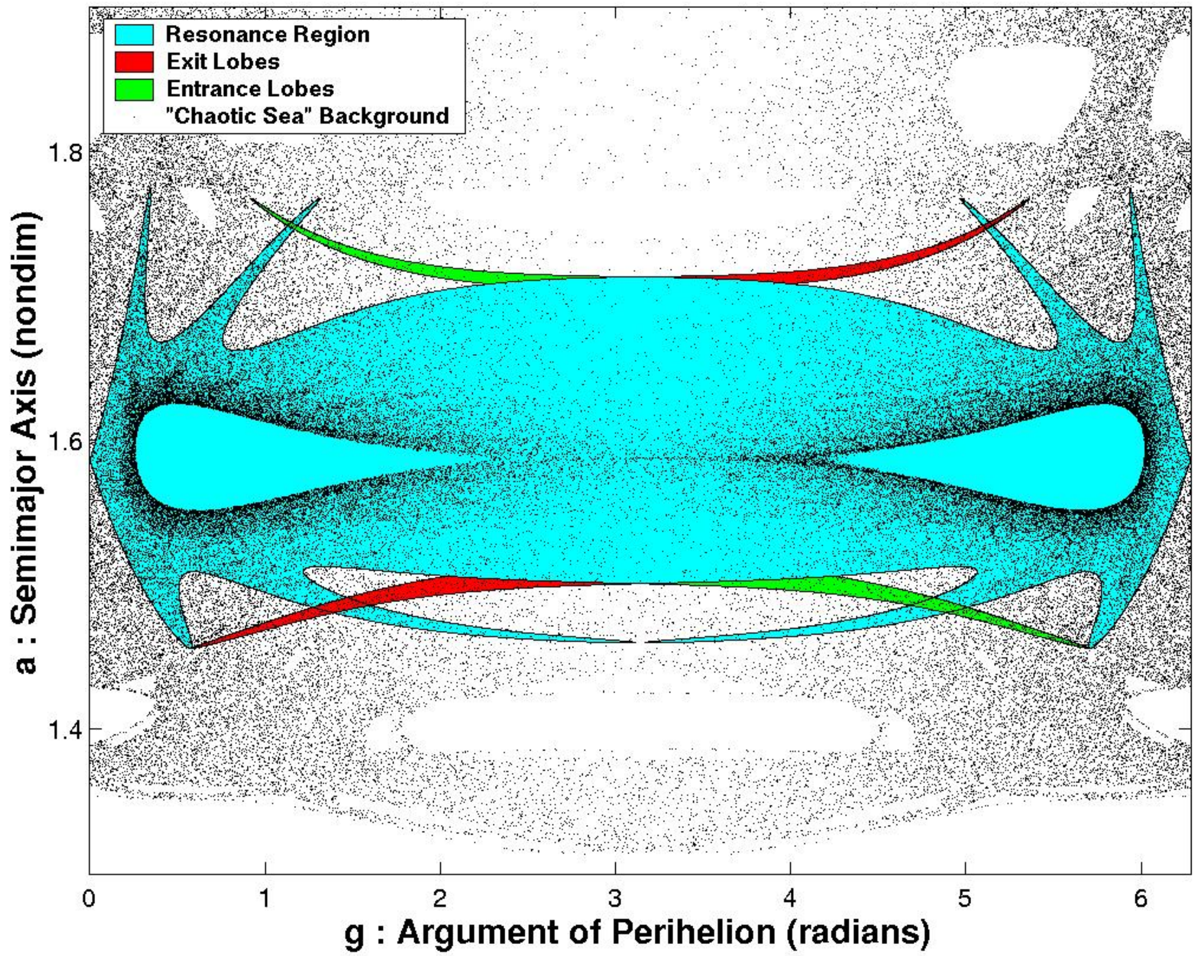
- The motion within the chaotic sea is understood as the movement of trajectories among resonance regions (see Meiss [1992] and Schroer and Ott [1997]).



Schematic of two neighboring resonance regions from Meiss [1992]

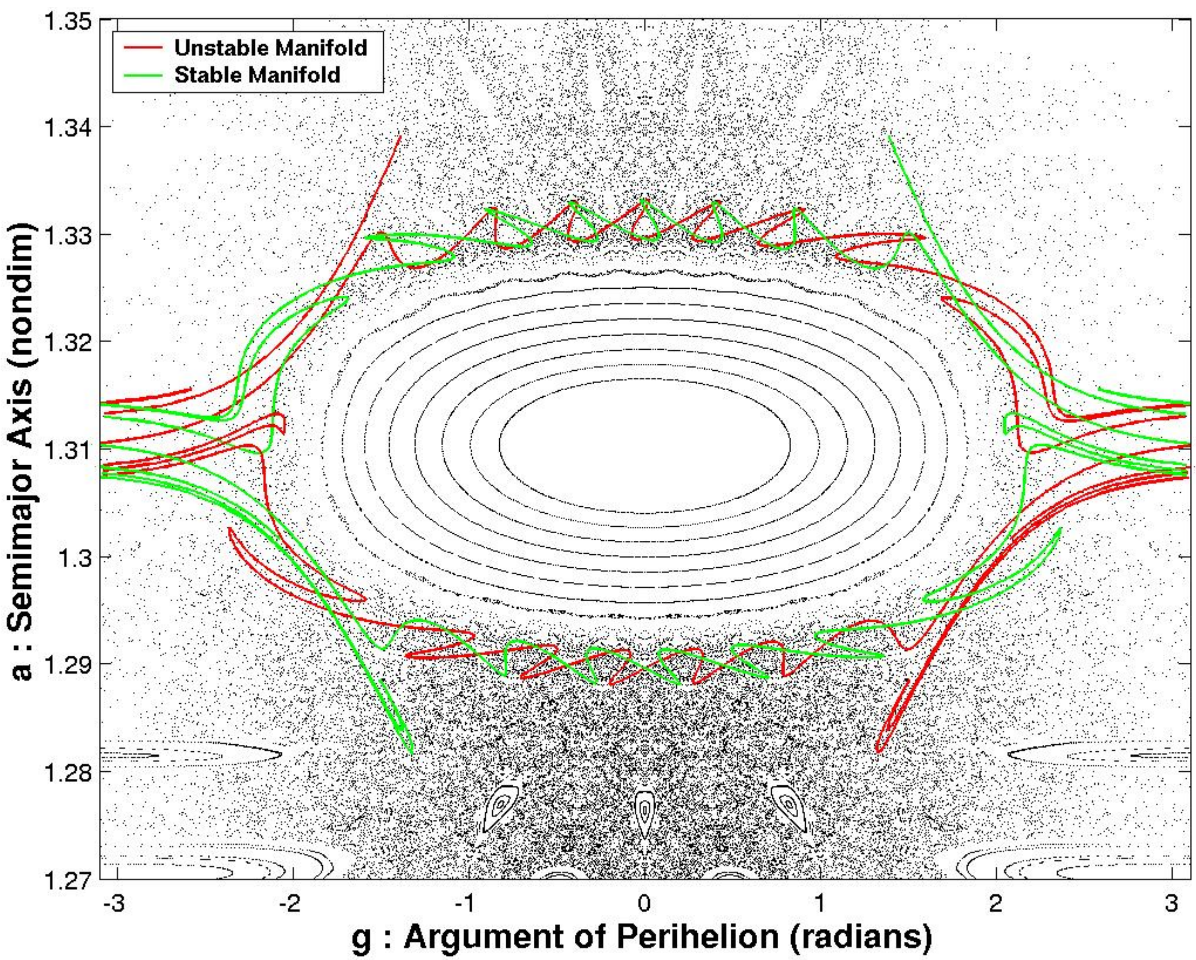
Movement among resonances

- This is confirmed by numerical computation.
- Shaded region bounded by stable and unstable invariant manifolds of an unstable resonant (periodic) orbit.



Movement among resonances

- The unstable and stable manifolds are understood as the backbone of the dynamics. This is the “homoclinic trellis” in the words of Poincaré.



Transport quantities

- There are several approaches to computing useful transport quantities.
  - Markov model where the energy shell is partitioned into stochastic regions separated by partial barriers (Meiss et al.)
  - Set oriented methods where a graph is created to model the underlying dynamical behavior (Dellnitz et al.)
  - Lobe dynamics; following intersections of stable and unstable invariant manifolds of periodic orbits (Wiggins et al.)
- These methods are preferred over the "brute force" astrodynamic calculations seen in the literature since they are based on first principles.

Transition Rates

- ***Fluxes give rates and probabilities***
  - Jaffé, Ross, Lo, Marsden, Farrelly, and Uzer [2002] computed the rate of escape of asteroids temporarily captured by Mars.
  - RRKM-like statistical approach
    - similar to chemical dynamics, see Truhlar [1996]
  - Consider an asteroid (or other body) in orbit around Mars (perhaps impact ejecta) at a 3-body energy such that it can escape toward the Sun.
  - Interested in rate of escape of such bodies at a fixed energy, i.e. $F_{M,S}(t)$

Transition Rates

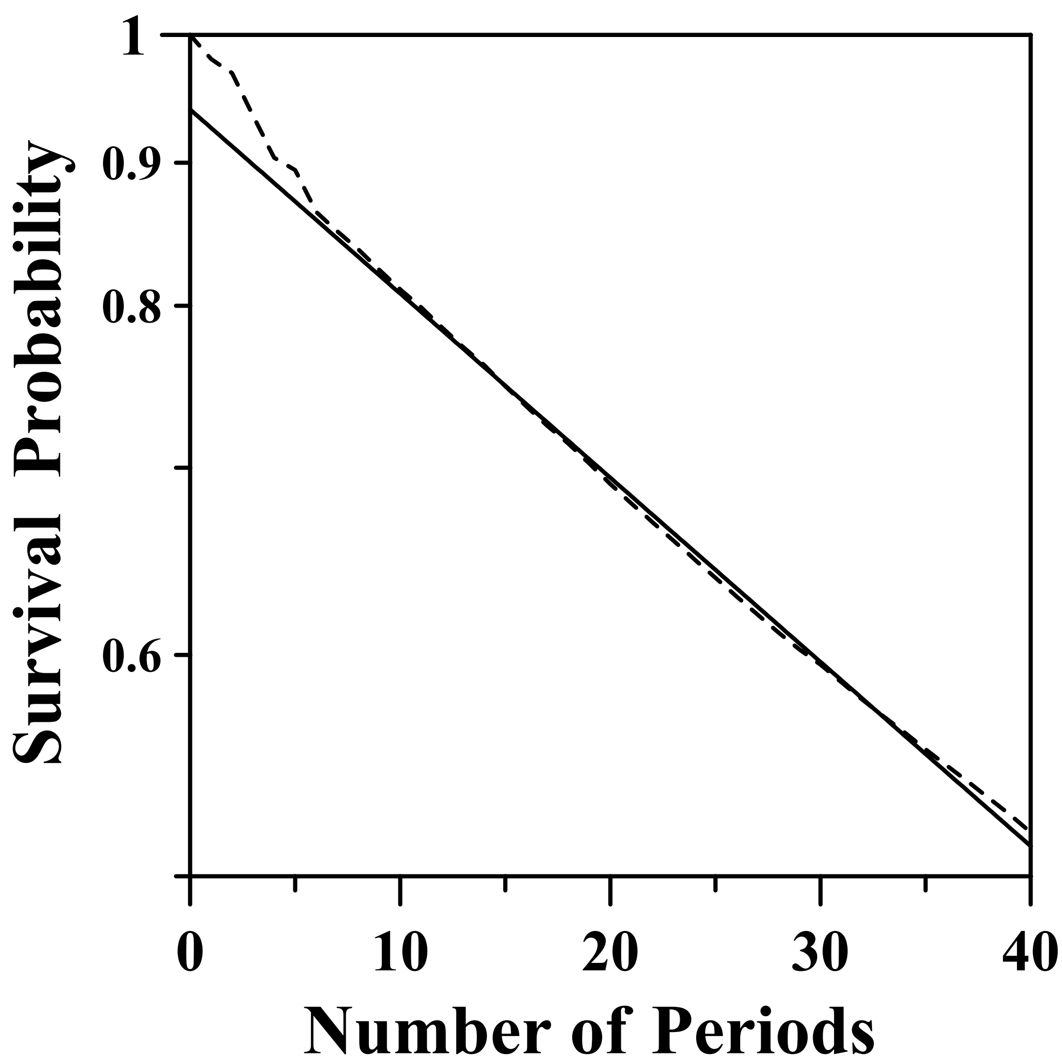
- RRKM assumption: all asteroids in the Mars region at fixed energy are **equally likely to escape**. Then

$$\text{Escape rate} = \frac{\text{flux across potential barrier}}{\text{Mars region phase space volume}}$$

- Compare with Monte Carlo simulations of 107,000 particles
  - randomly selected initial conditions at constant energy

Transition Rates

- Theory and numerical simulations agree well.
  - Monte Carlo simulation (dashed) and theory (solid)



Steady state distribution

- If the planar, circular restricted three-body problem is **ergodic,** then a statistical mechanics can be built (cf. ZhiGang [1999]).
- Recent work suggests there may be regions of the energy shell for which the motion is ergodic, in particular the "chaotic sea" (Jaffé et al. [2002]).
- This suggests we compute the **steady state distribution** of some observable for particles in the chaotic sea; a simple method for obtaining the likely locations of any particles within it.

Steady state distribution

- Assuming ergodicity,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t A(x,y,p_x,p_y)d\tau =$$

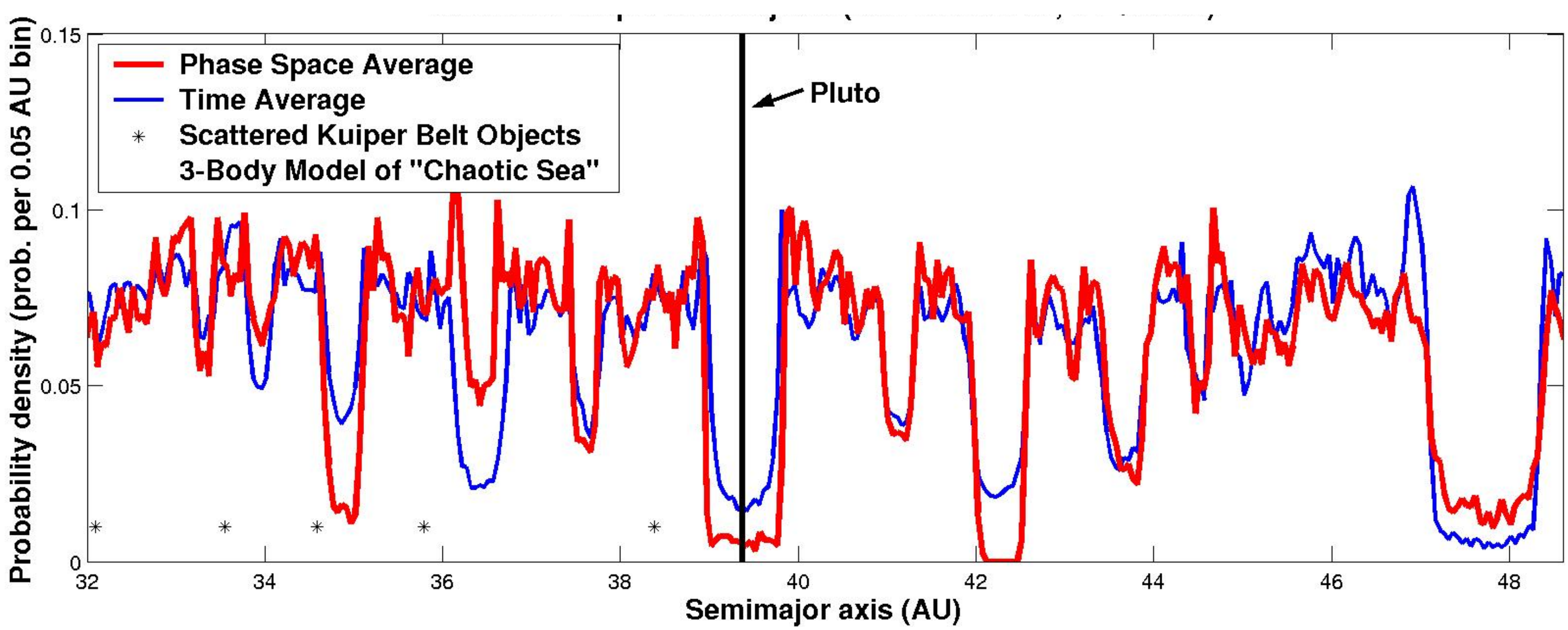
$$\int A(x,y,p_x,p_y)\,\frac{C}{|\frac{\partial H}{\partial p_y}|}dp_x dx dy,$$

  where $A(x,y,p_x,p_y)$ is any physical observable (e.g., semimajor axis), one finds that the density function, $\rho(x,p_x)$, on the surface-of-section, $\Sigma_{(\mu,\varepsilon)}$, is constant.

- We can determine the steady state distribution of semi-major axes; define $N(a)da$ as the number of particles falling into $a \rightarrow a+da$ on the surface-of-section, $\Sigma_{(\mu,\varepsilon)}$.

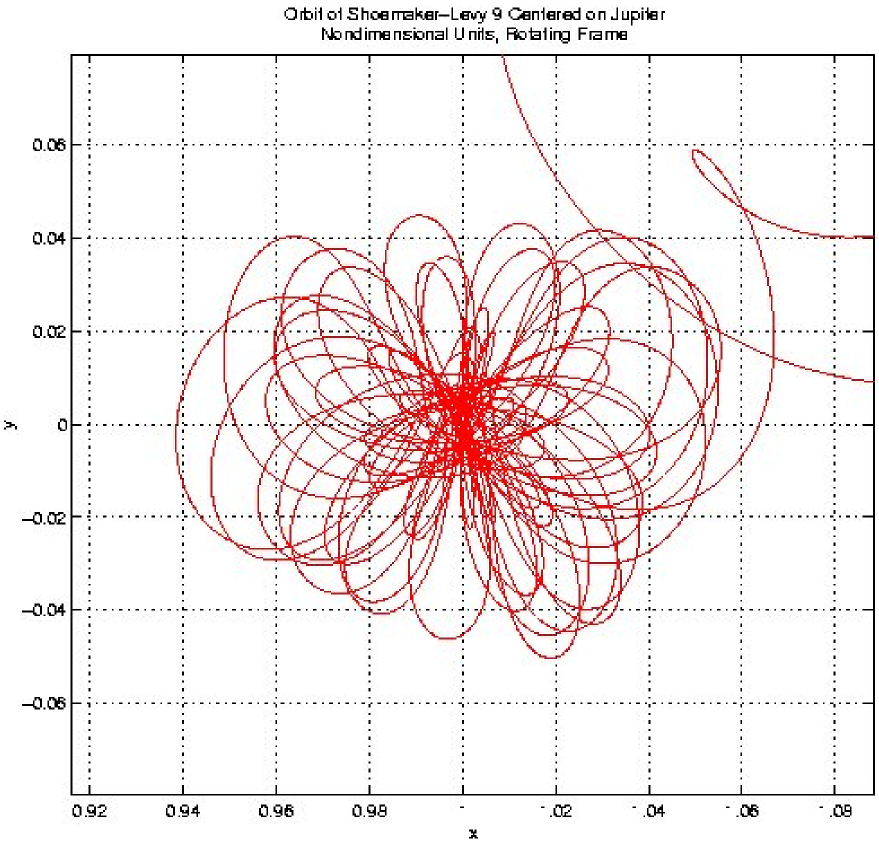
Steady state distribution

- SKBOs should be in regions of high density.



Collisions with Jupiter

- **Shoemaker Levy-9**: similar energy to **Oterma**
  - Temporary capture and collision; came through L1 or L2

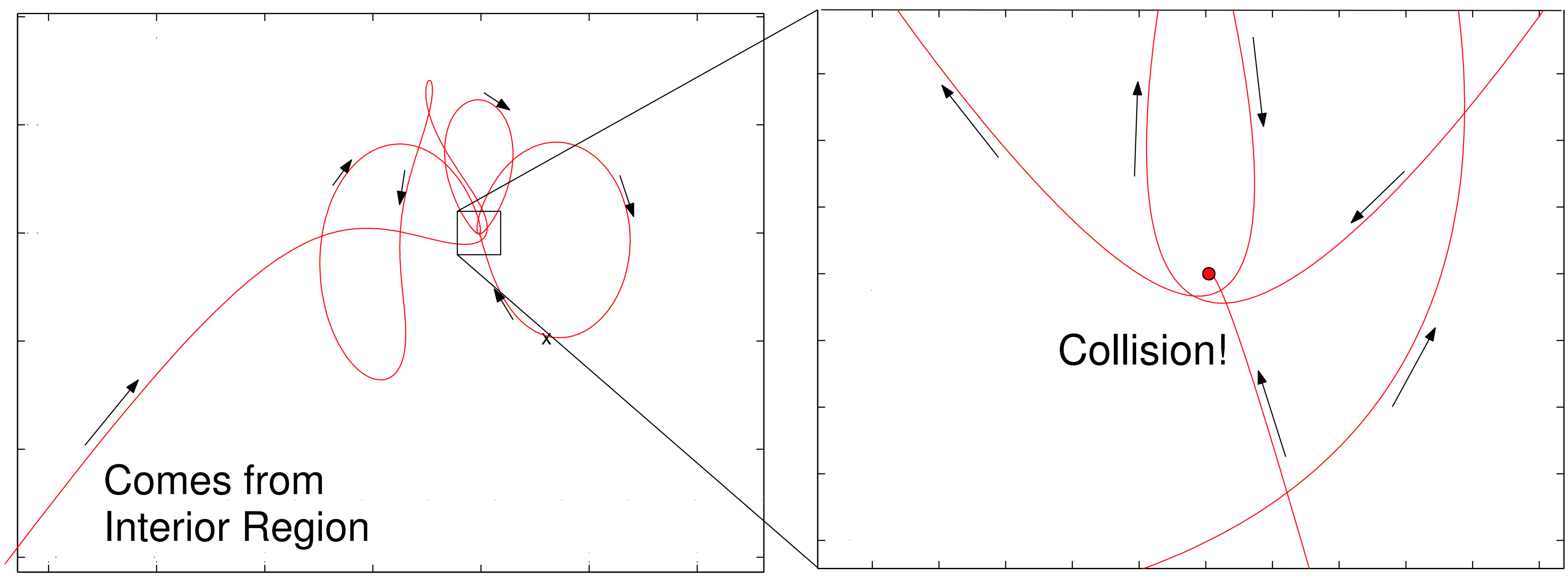


Possible *Shoemaker-Levy 9* orbit seen in rotating frame (Chodas, 2000)

Collision Probabilities

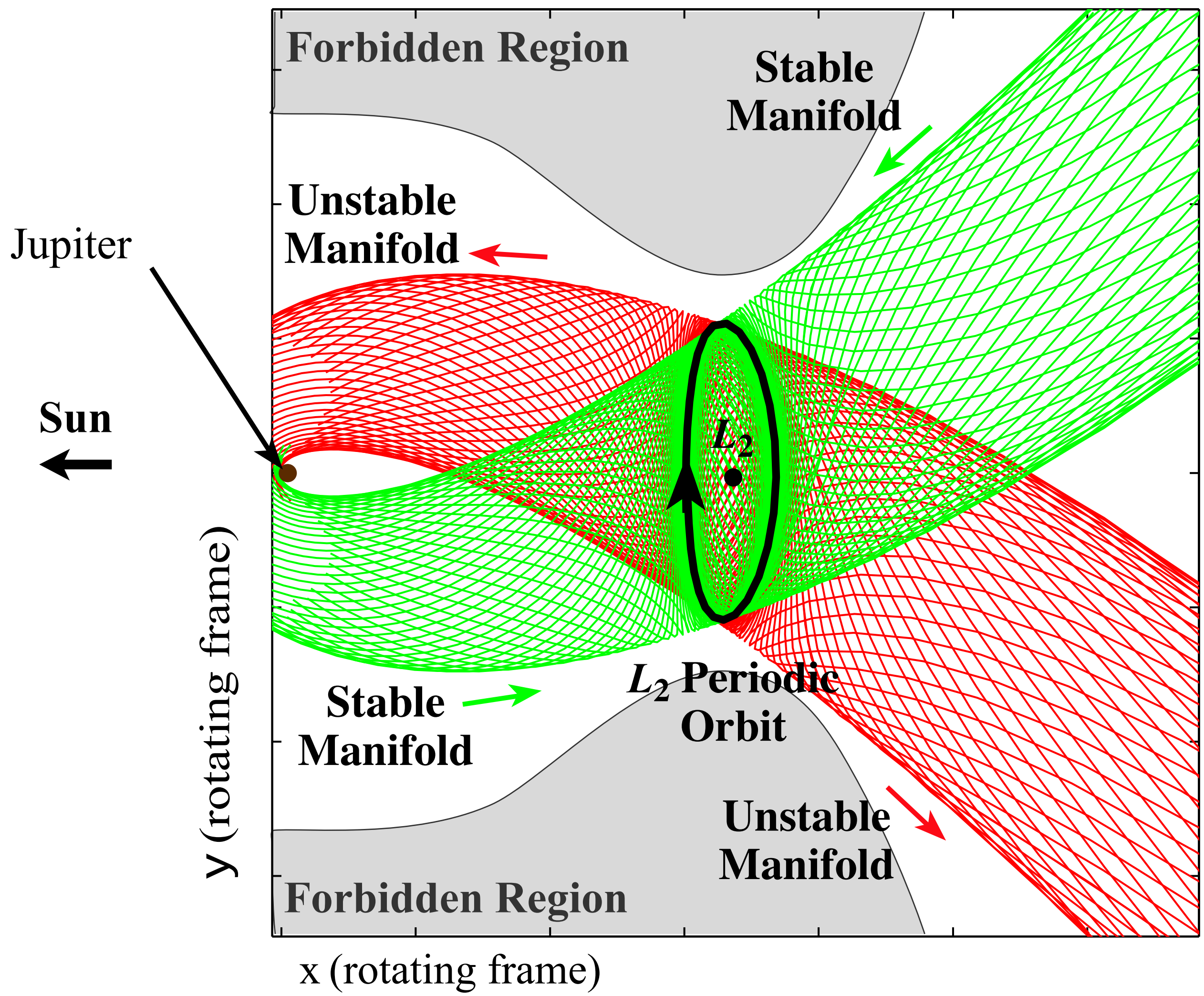
- Low velocity impact probabilities
- Assume object enters the planetary region with an energy slightly above L1 or L2
  - eg, **Shoemaker-Levy 9** and **Earth-impacting asteroids**

Example Collision Trajectory



Tubes in the 3-Body Problem

- **Stable** and **unstable** **manifold tubes**
  - Control transport through the potential barrier.



Collision Probabilities

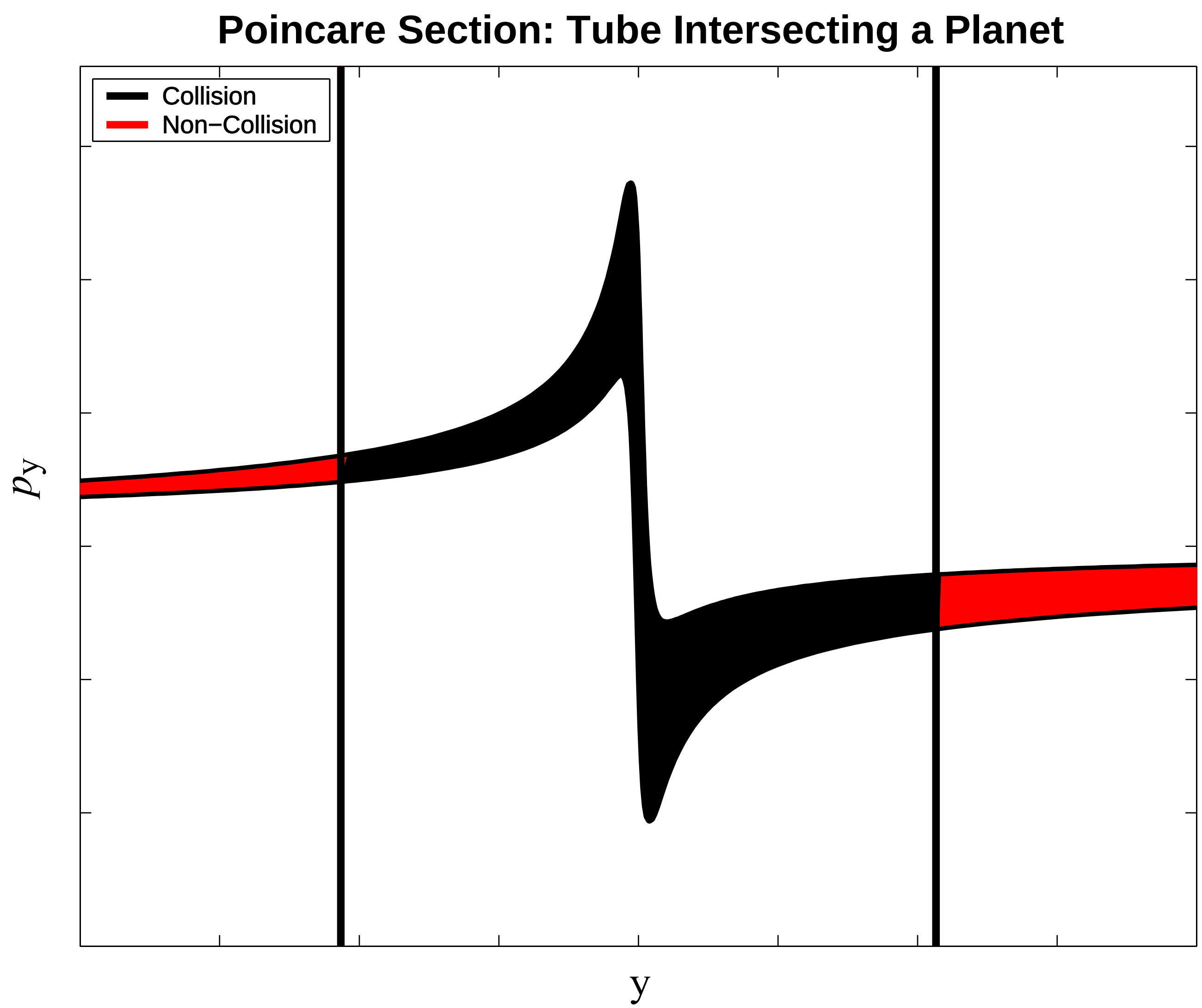
Collision probabilities

- Compute from tube intersection with planet on Poincaré section
- Planetary diameter is a parameter, in addition to μ and energy E

$\longleftarrow$ Diameter of planet $\longrightarrow$

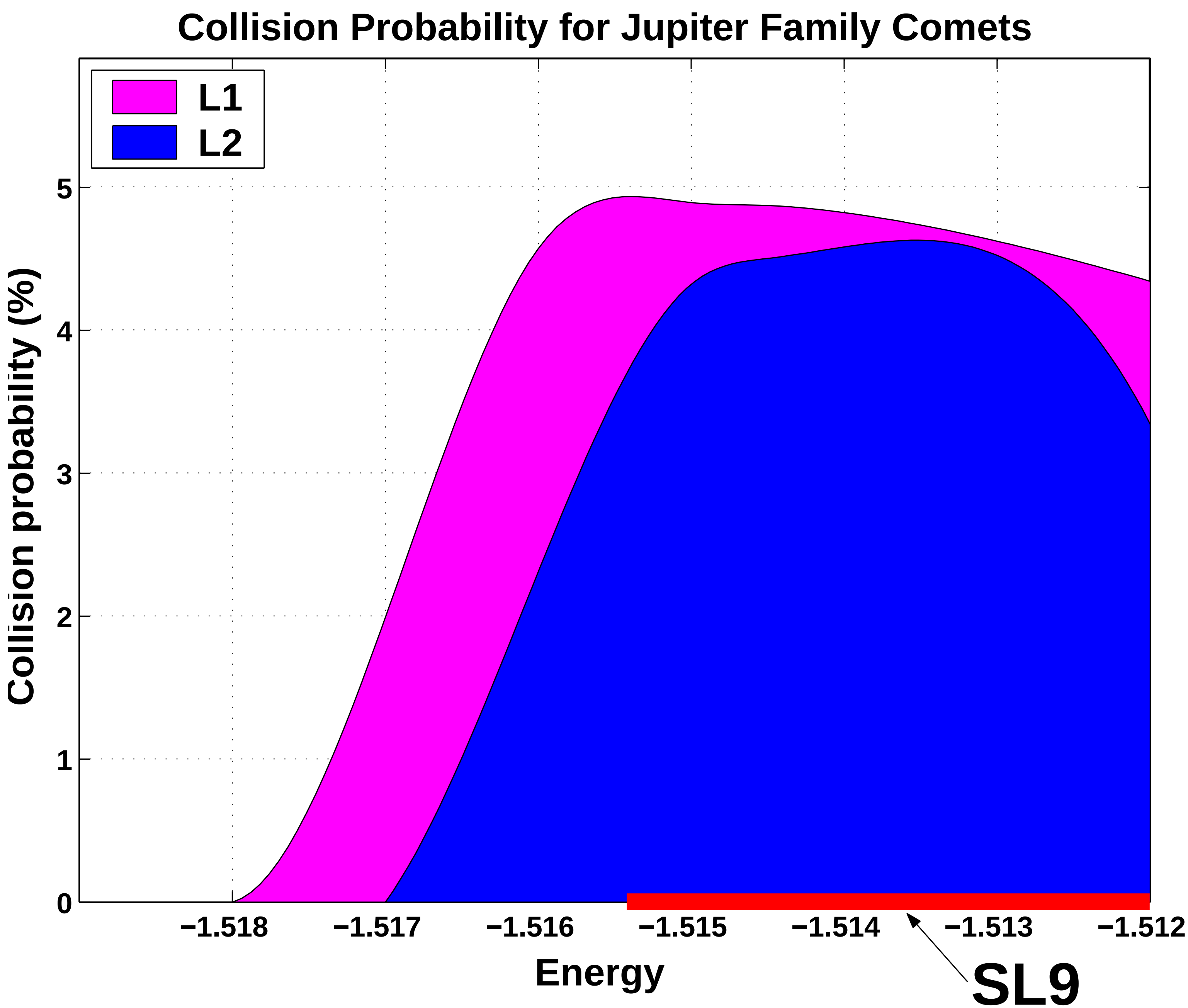
Collision Probabilities

■ *Collision probabilities*

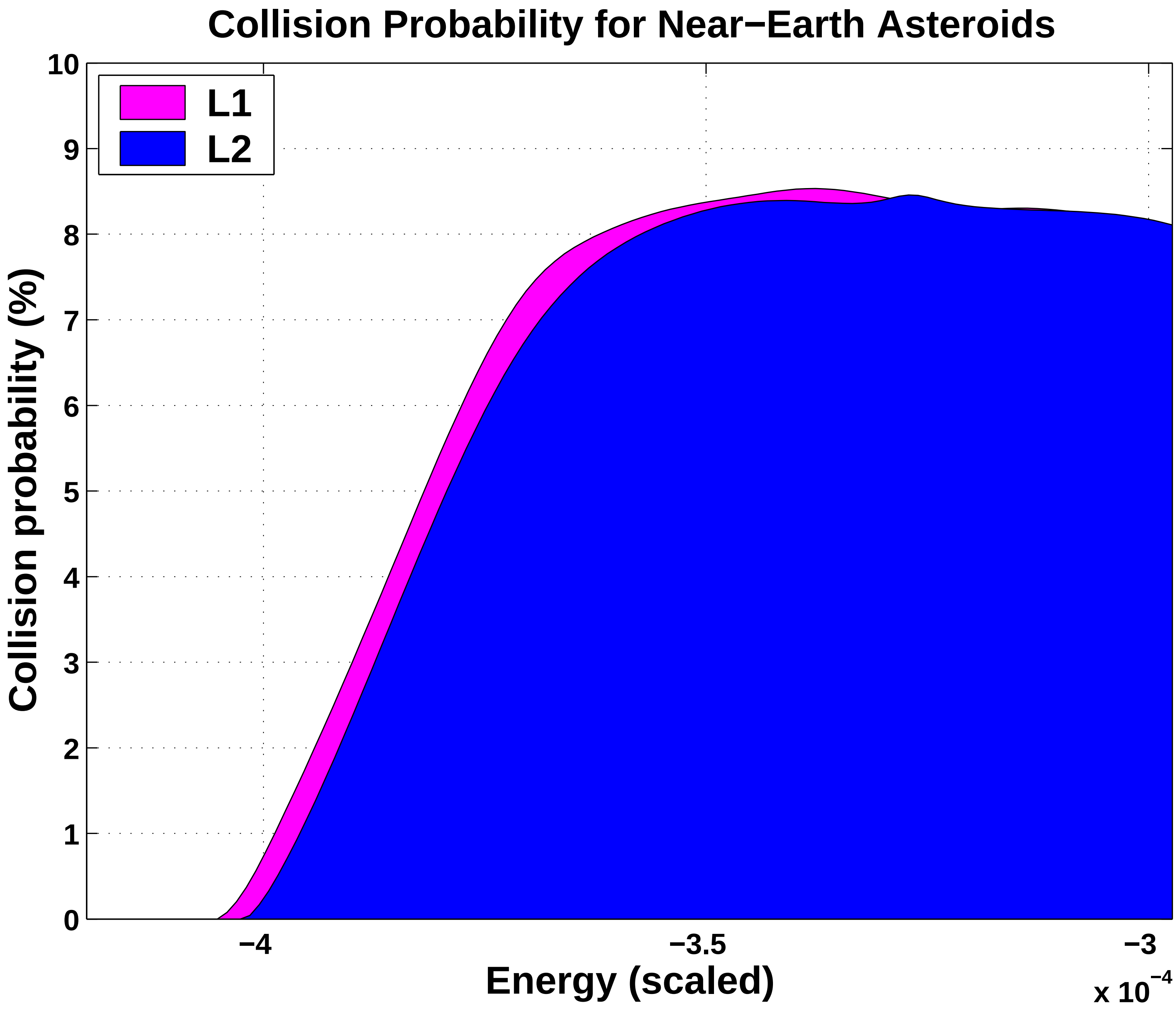


← Diameter of planet →

Collision Probabilities



Collision Probabilities



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The End