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Resonance Overlap and Transport in the Restricted Three-Body Problem

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Outline of talk

- ***Insight into some dynamical astronomy phenomena can be gained by a restricted three-body analysis***
  - e.g., Jupiter-family comets and scattered Kuiper Belt objects (under Neptune's control); near-Earth objects
  - By applying dynamical systems methods to the planar, circular restricted three-body problem, several questions regarding these populations may be addressed
  - Outline some theoretical ideas
  - Several computational results will be shown
  - Comparison with observational data is made
  - Future directions: other N-particle systems

Transport Theory

- ***Chaotic dynamics***
  $\Longrightarrow$ ***statistical methods***
- ***Transport theory***
  - Ensembles of phase space trajectories
    - How long (or likely) to move from one region to another?
    - Determine transition probabilities, correlation functions
  - Applications:
    - Atomic ionization rates
    - Chemical reaction rates
    - Comet and asteroid escape rates, resonance transition probabilities, collision probabilities

Transport Theory

Transport in the solar system

- For objects of interest
  - e.g., Jupiter family comets, near-Earth asteroids, dust
- **Identify phase space objects** governing transport
- View N-body as multiple restricted 3-body problems
- Look at stable and unstable manifold of periodic orbits associated with Lagrange points and mean motion resonances
- Use these to **compute statistical quantities**
  - e.g., probability of resonance transition, escape rates

Dynamical astronomy

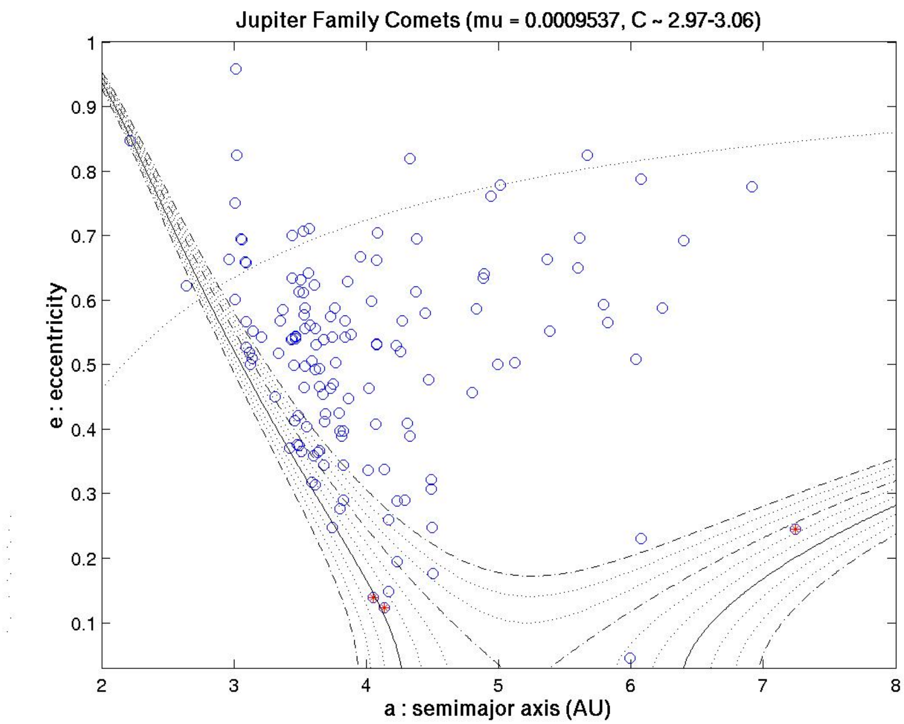
- We want to answer several questions regarding the transport and origin of some kinds of solar system material
  - How do we characterize the motion of Jupiter-family comets (JFCs) and scattered Kuiper Belt objects (SKBOs)?
  - How probable is a Shoemaker-Levy 9-type collision with Jupiter? Or an asteroid collision with Earth (e.g., KT impact 65 Ma)?
  - How likely is a transition from outside a planet's orbit to inside (e.g., the dance of comet Oterma with Jupiter)?
  - We can answer these questions by considering the phase space
- Harder questions
  - How does impact ejecta get from Mars to Earth?
  - How does an SKBO become a comet or an Oort Cloud comet?
  - Find features common to all exo-solar planetary systems?

Jupiter Family Comets

- JFCs and lines of constant **Tisserand parameter**,

$$T = \frac{1}{a} + 2\sqrt{a(1-e^2)},$$

an approximation of the Jacobi constant (i.e., $C = T + \mathcal{O}(\mu)$)

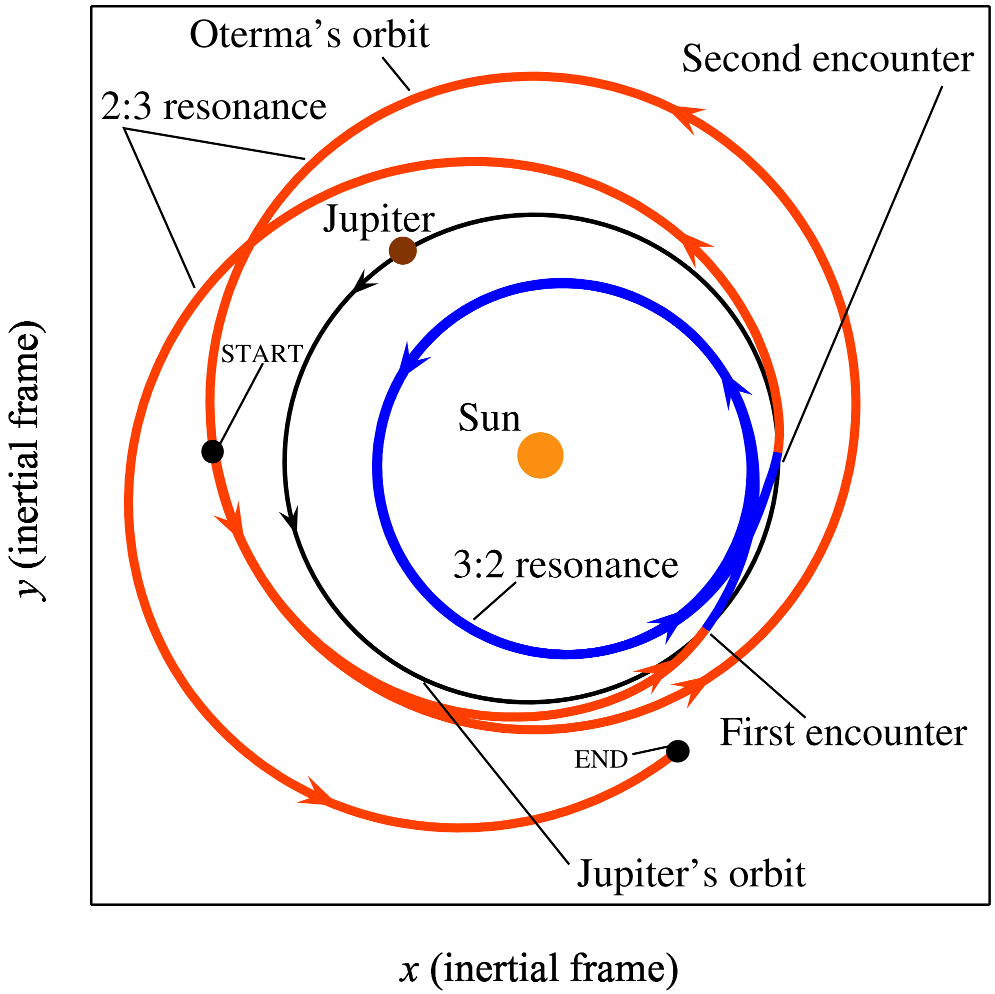


Jupiter Family Comets

- *Physical example of intermittency*
  - We consider the **historical record** of the comet **Oterma** from 1910 to 1980
    - first in an inertial frame
    - then in a rotating frame
    - a special case of pattern evocation
  - Similar pictures exist for many other comets

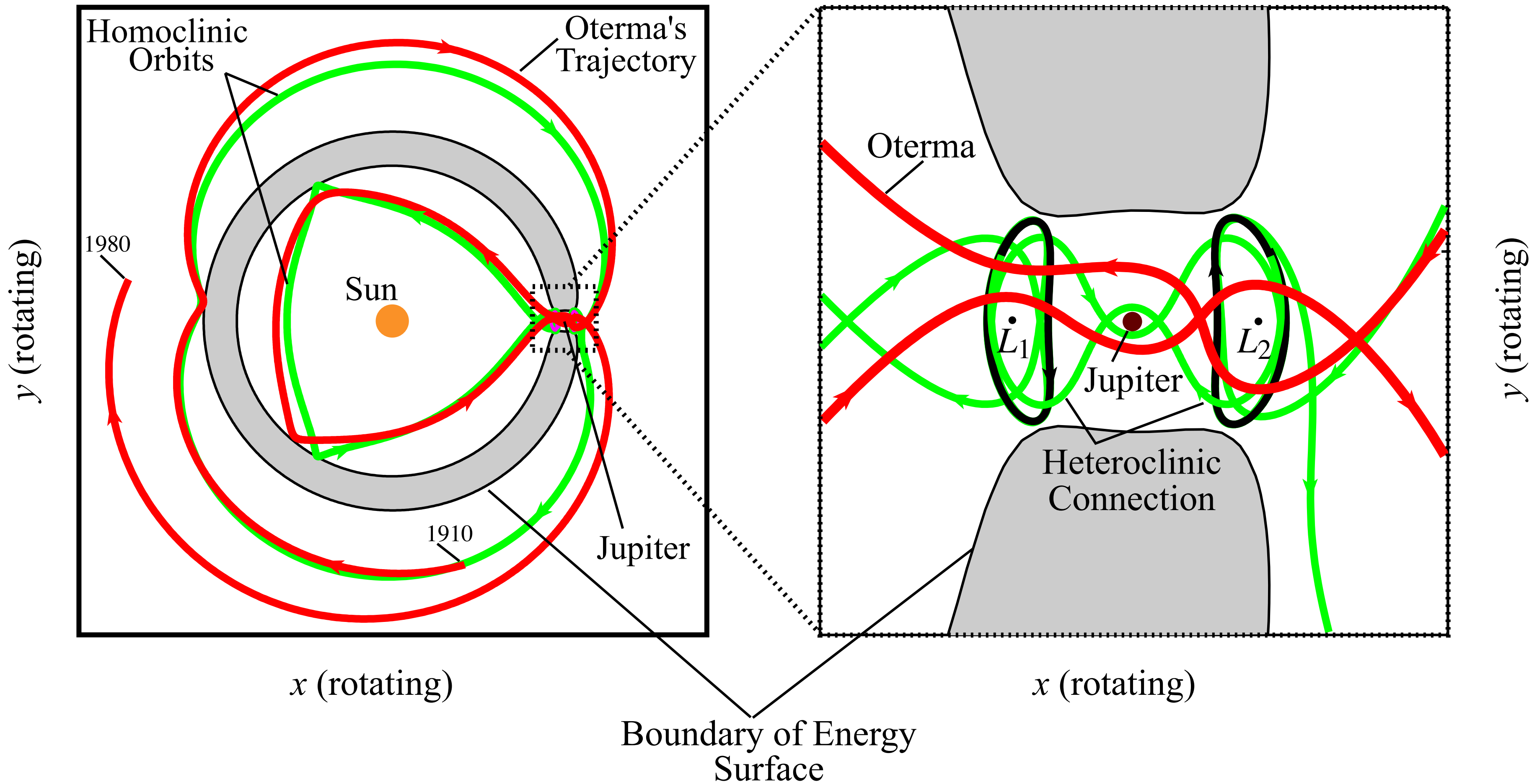
Jupiter Family Comets

- Rapid transition: outside to inside Jupiter's orbit.
  - Captured temporarily by Jupiter during transition.
  - Exterior (2:3 resonance) to interior (3:2 resonance).



Viewed in Rotating Frame

- **Oterma**'s orbit in rotating frame with some invariant manifolds of the 3-body problem superimposed.

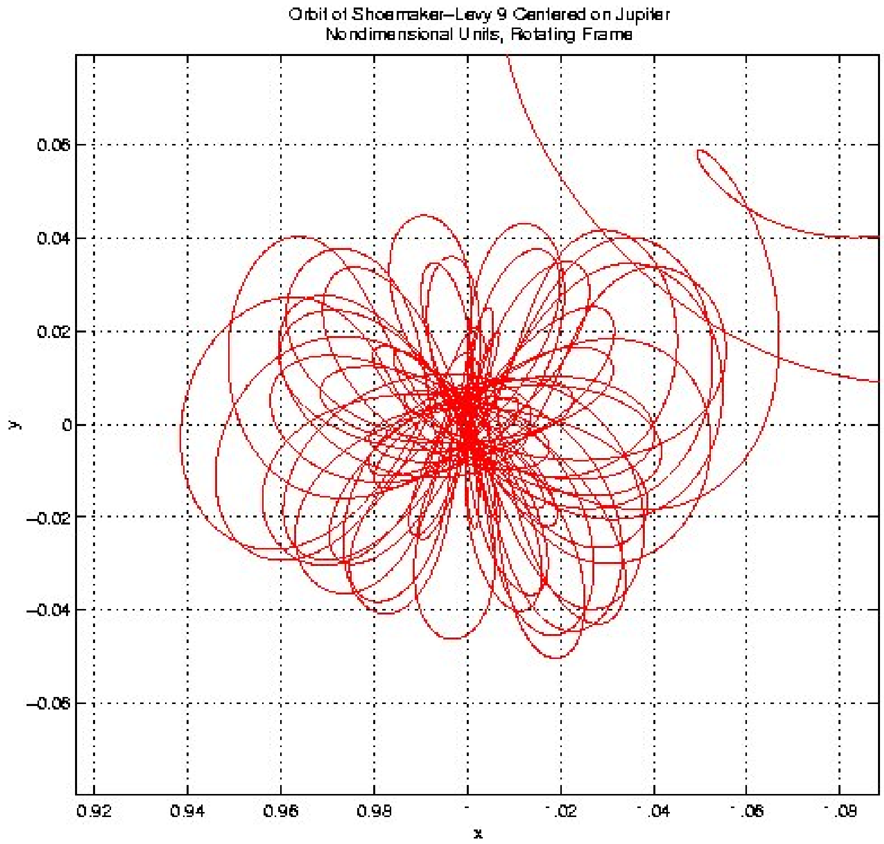


Viewed in Rotating Frame

Oterma - Rotating Frame

Collisions with Jupiter

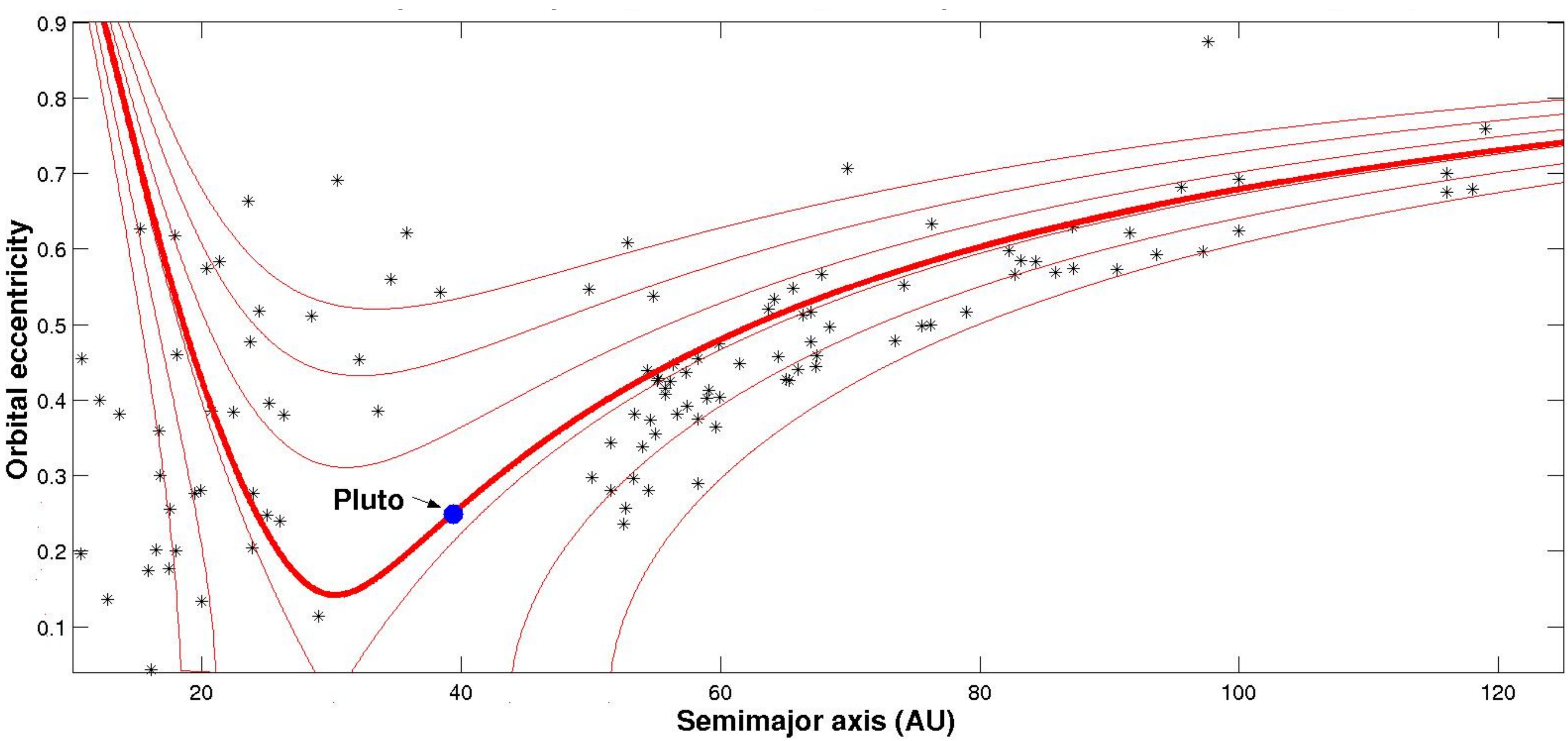
- **Shoemaker Levy-9**: similar energy to **Oterma**
  - Temporary capture and collision; came through L1 or L2

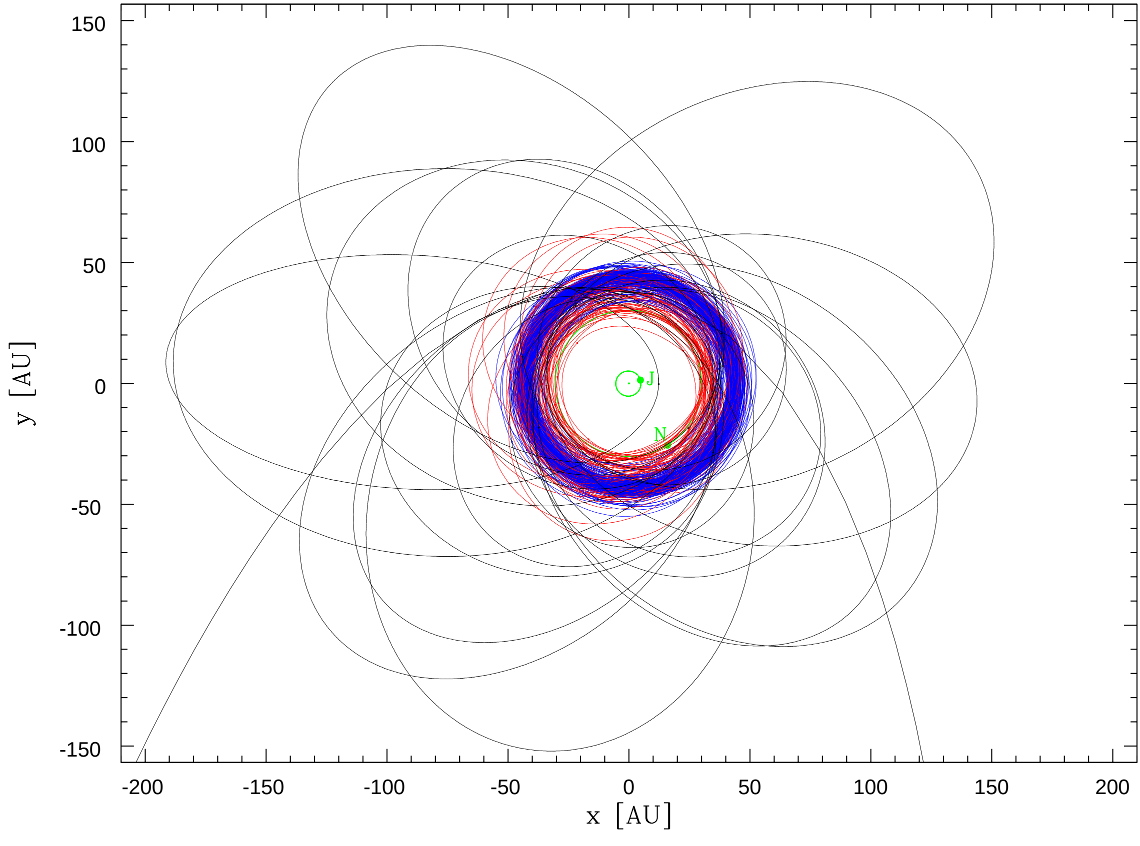


Possible *Shoemaker-Levy 9* orbit seen in rotating frame (Chodas, 2000)

Scattered Kuiper Belt objects

- Current SKBO locations in black, with some Tisserand values w.r.t. Neptune in red ($T \approx 3$)





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y [AU]
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Motion of JFCs and SKBOs

- Observation and numerical experiments show chaotic motion maintaining nearly constant Tisserand parameter in the short-term (i.e., a few Lyapunov times, $\sim 10^2$ to 10^3 years, cf. Tancredi [1995])
- We approximate the short-timescale motion of JFCs and SKBOs as occurring within an energy shell of the restricted three-body problem
- Several objects may be in nearly the same energy shell, i.e., all have T s.t. $|T - T^*| \leq \delta T$ for some $T^*, \delta T$
- We analyze the structure of an energy shell to determine likely locations of JFCs and SKBOs

Three-Body Problem

Circular restricted 3-body problem

- the two primary bodies move in circles; the much smaller third body moves in the gravitational field of the primaries, without affecting them
- the two primaries could be the Sun and Earth, the Earth and Moon, or Jupiter and Europa, etc.
- the smaller body could be a spacecraft or asteroid
- we consider the planar and spatial problems
- there are five equilibrium points in the rotating frame, places of balance which generate interesting dynamics

Three-Body Problem

- Equations of motion:

$$\boxed{\ddot{x} - 2\dot{y} = -U_x^{\text{eff}}, \qquad \ddot{y} + 2\dot{x} = -U_y^{\text{eff}}}$$

  where

$$U^{\text{eff}} = -\frac{(x^2 + y^2)}{2} - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}.$$

- Have a first integral, the Hamiltonian energy, given by

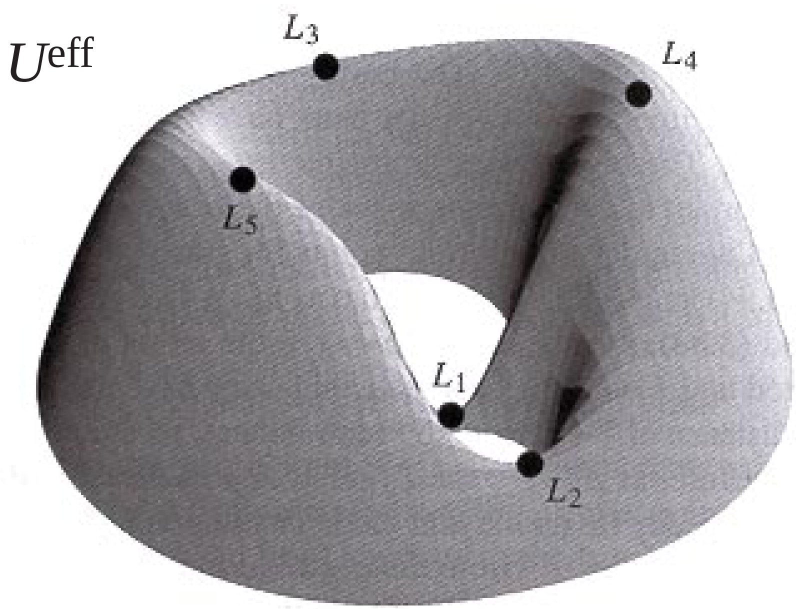
$$E(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U^{\text{eff}}(x, y).$$

- Energy manifolds are 3-dimensional surfaces foliating the 4-dimensional phase space.
- This is for the planar problem, but the spatial problem is similar.

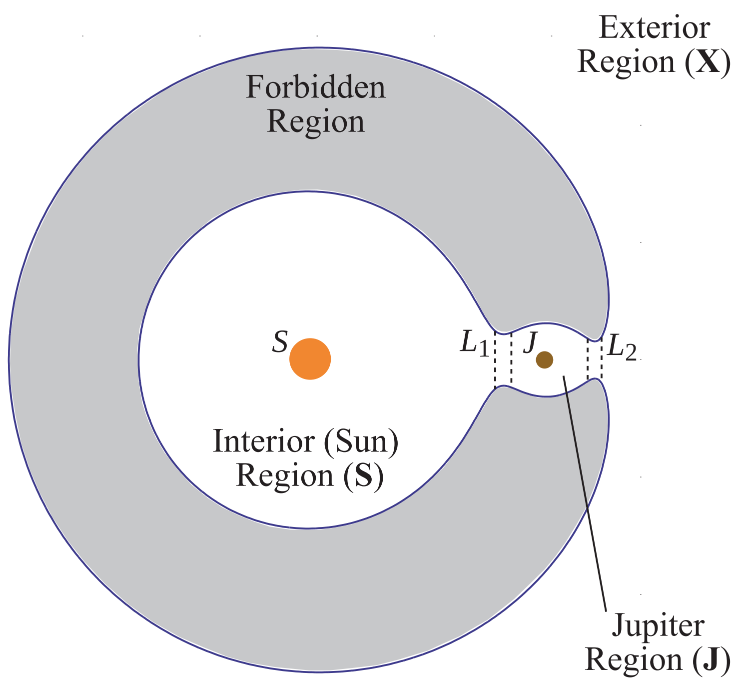
Regions of Possible Motion

Effective potential

- In a rotating frame, the equations of motion describe a particle moving in an effective potential plus a coriolis force (goes back to the work of Jacobi, Hill, etc.)



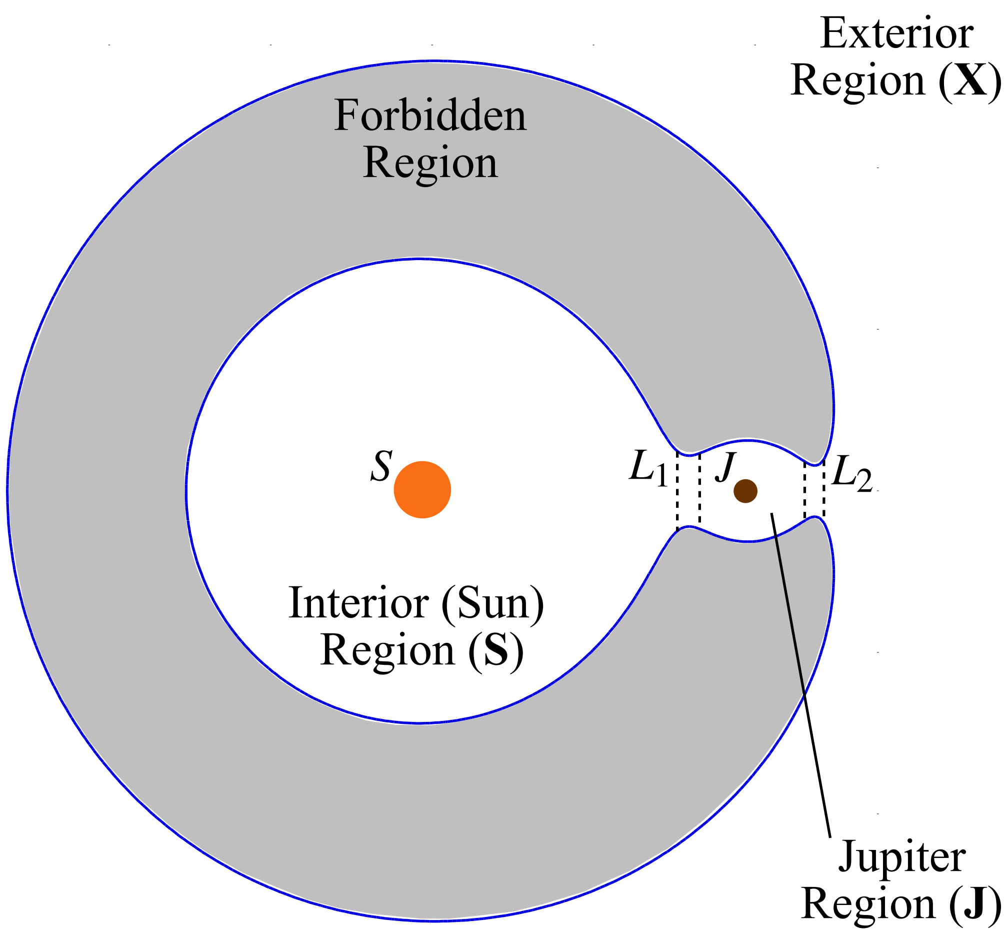
Effective potential



Level set shows accessible regions

Partition the Energy Surface

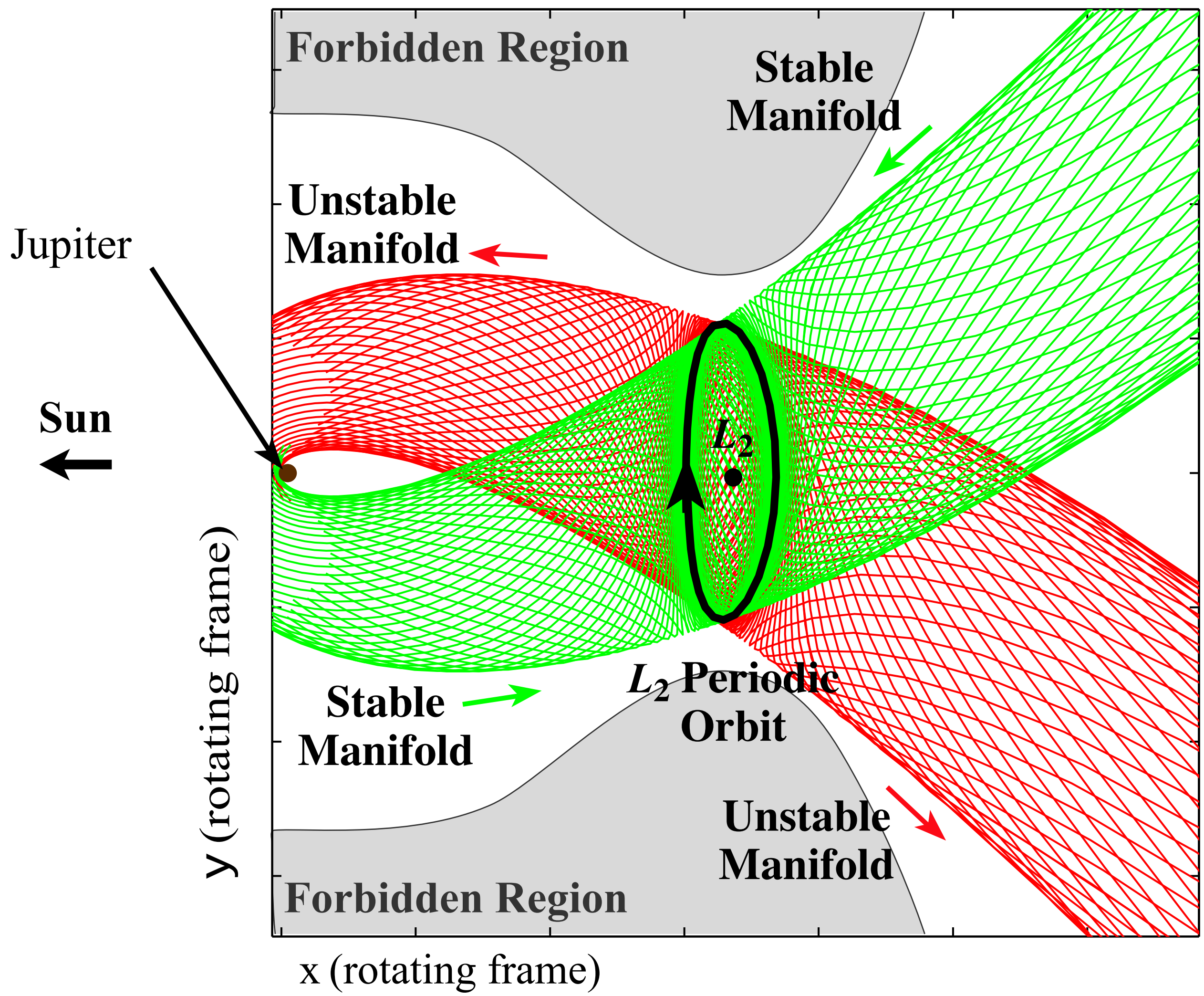
- ***Restricted 3-body problem (planar)***
  - Partition the energy surface: **S, J, X regions**



Position Space Projection

Tubes in the 3-Body Problem

- **Stable** and **unstable** **manifold tubes**
  - Control transport through the neck.



Motion within energy shell

- For fixed μ, an energy shell (or energy manifold) of energy ε is

$$\mathcal{M}(\mu,\varepsilon) = \{(x,y,\dot{x},\dot{y}) \mid E(x,y,\dot{x},\dot{y}) = \varepsilon\}.$$

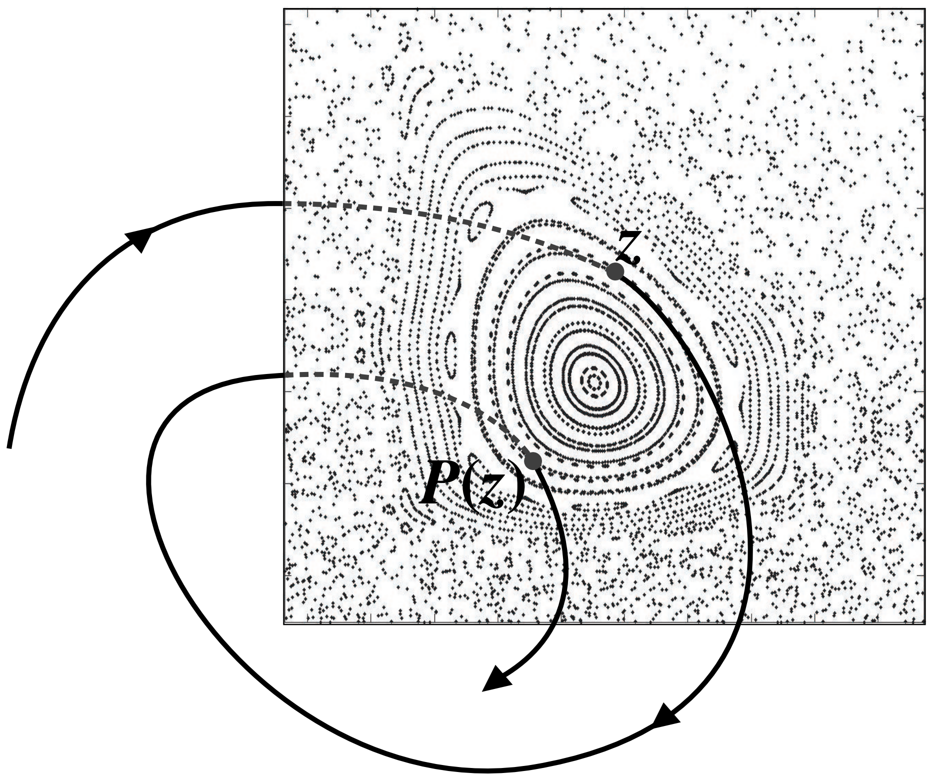
The $\mathcal{M}(\mu,\varepsilon)$ are 3-dimensional surfaces foliating the 4-dimensional phase space.

Poincaré surface-of-section

- Study Poincaré surface of section at fixed energy ε:

$$\Sigma_{(\mu,\varepsilon)} = \{(x,\dot{x})|y=0, \dot{y}=f(x,\dot{x},\mu,\varepsilon)<0\}$$

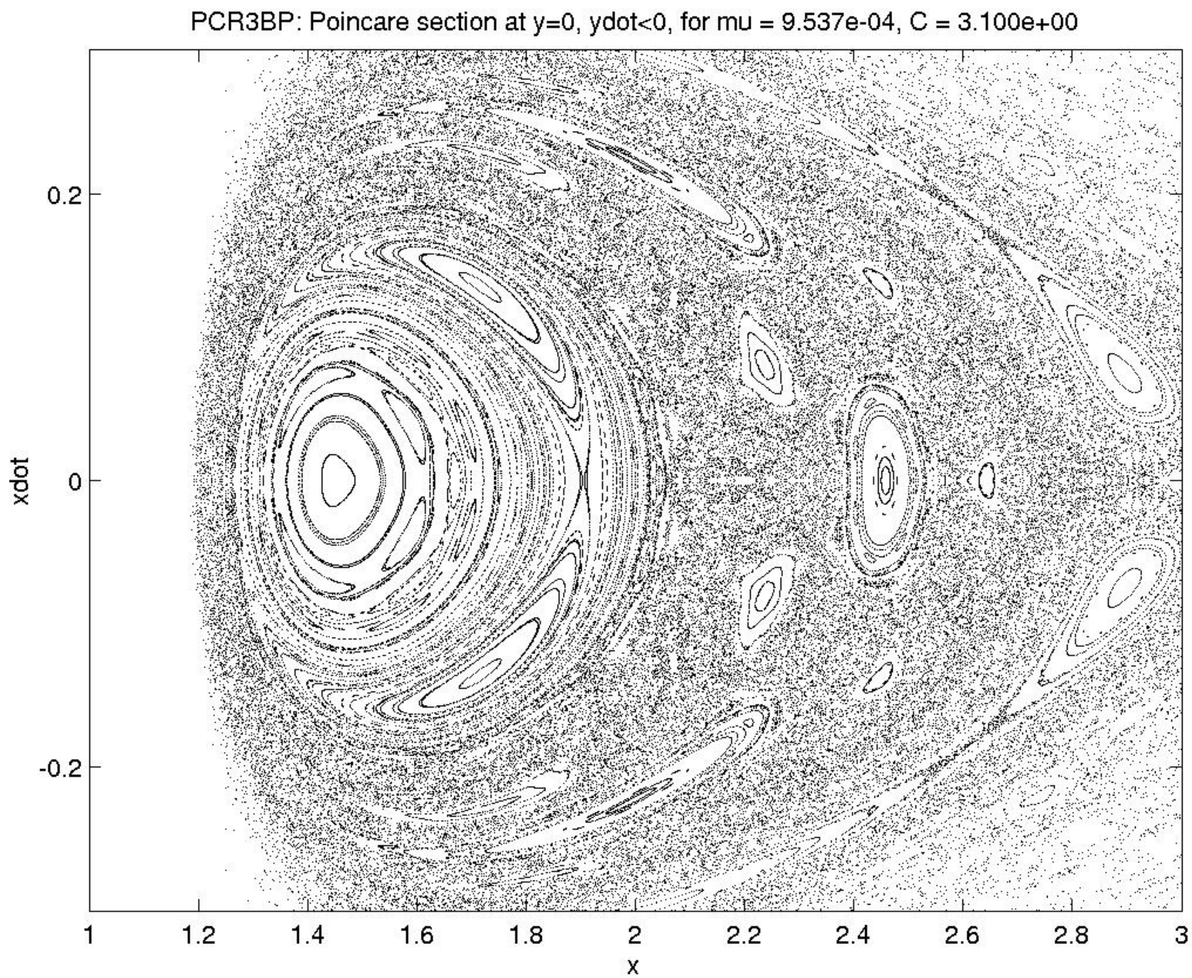
reducing the system to an area preserving map on the plane. Motion takes place on the cylinder, $S^1 \times \mathbb{R}$.



Poincaré surface-of-section and map P

Connected chaotic component

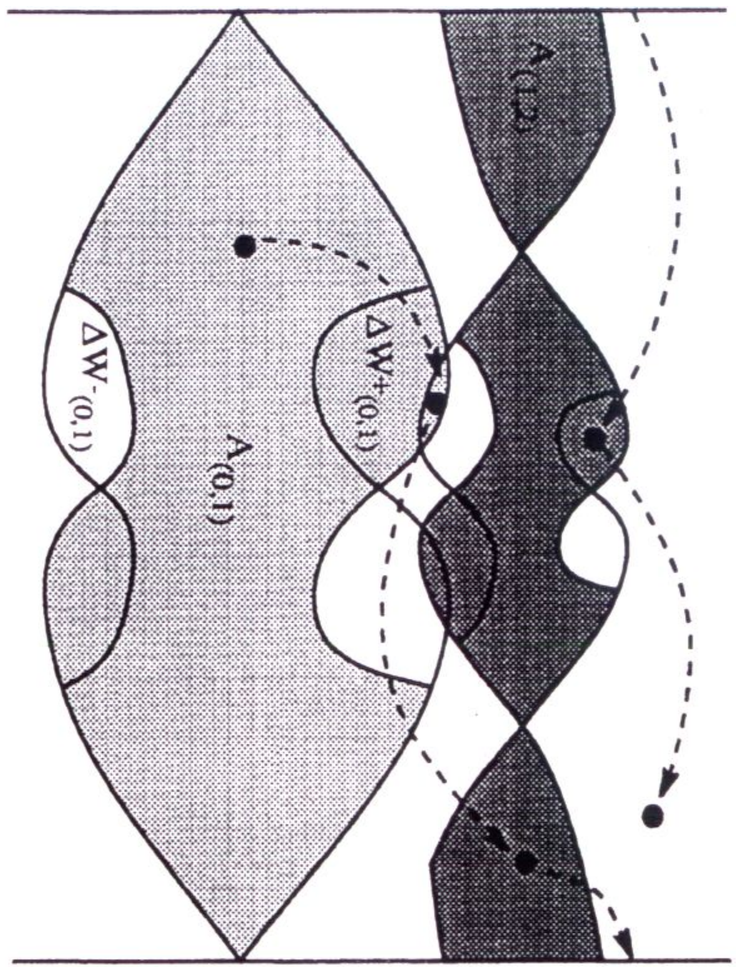
- The energy shell has regular components (KAM tori) and irregular components. Large connected irregular component is the **"chaotic sea."**



Movement among resonances

- The motion within the chaotic sea is understood as the movement of trajectories among resonance regions (see Meiss [1992] and Schroer and Ott [1997]).

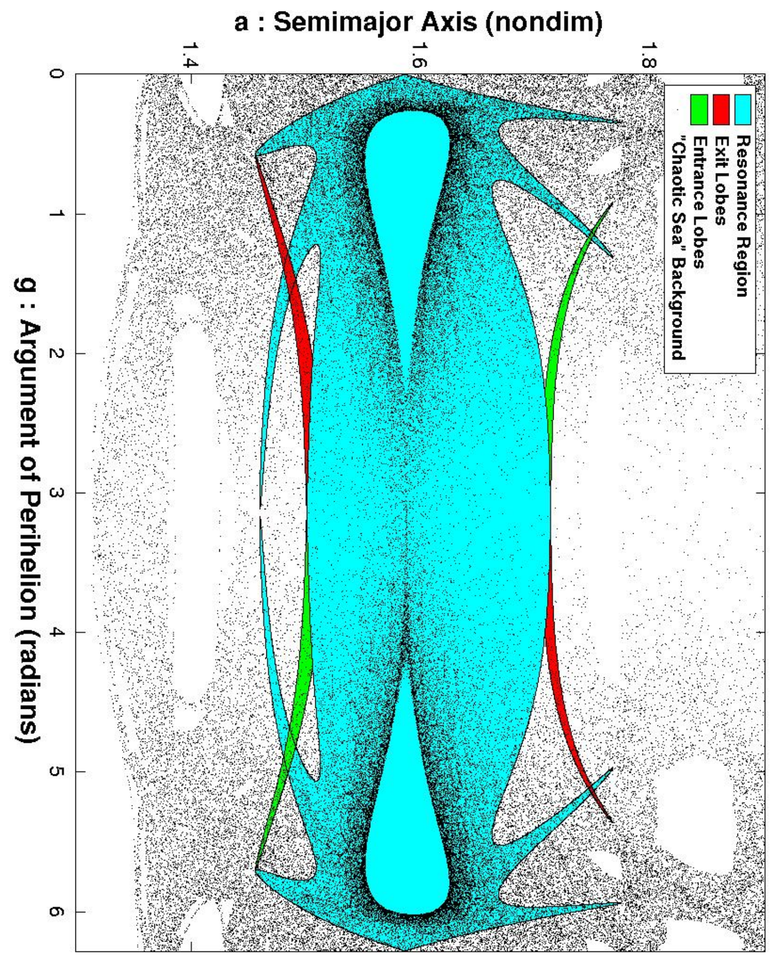
Movement among resonances



Schematic of two neighboring resonance regions from Meiss [1992]

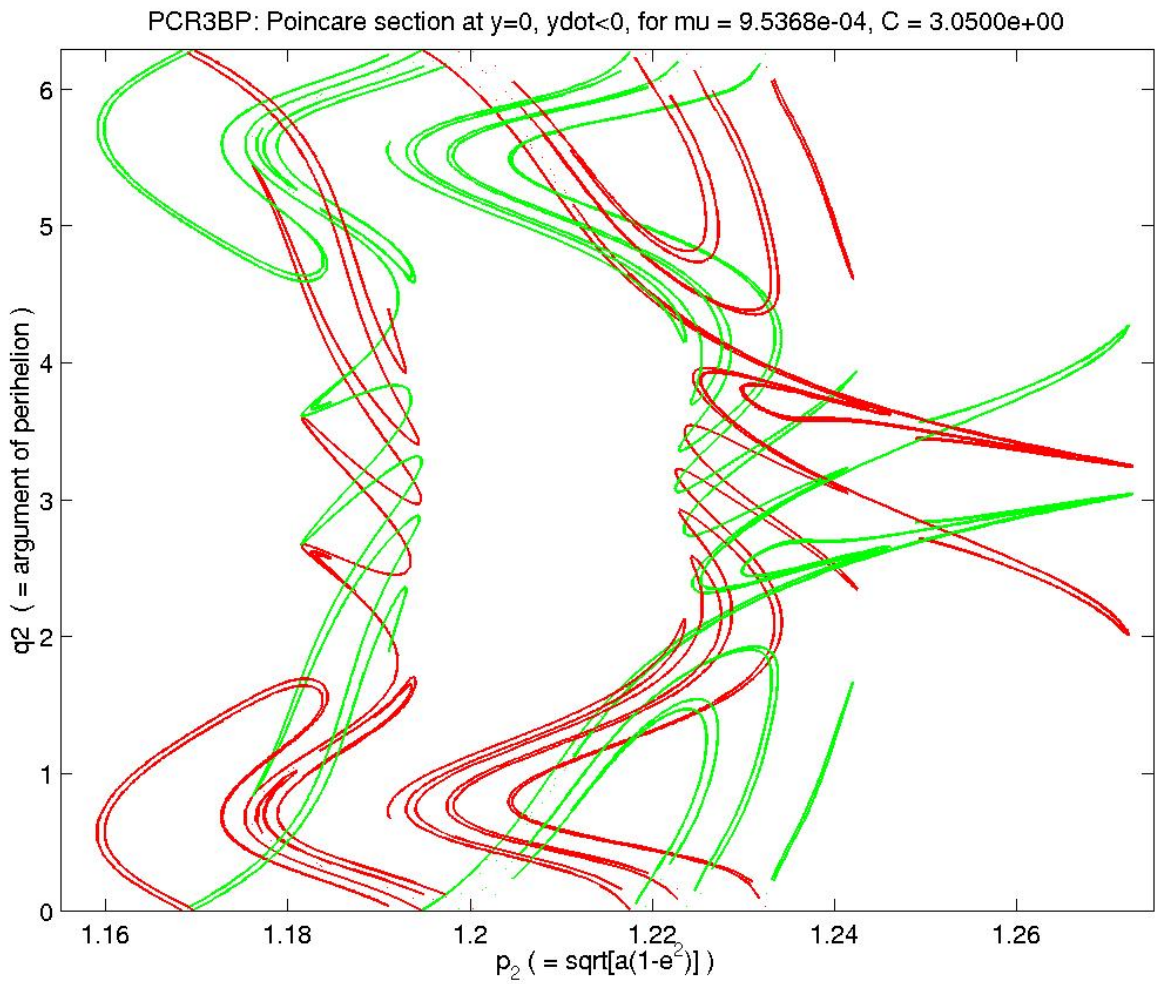
Movement among resonances

- Confirmed by numerical computation.
- Shaded region bounded by stable and unstable invariant manifolds of an unstable resonant (periodic) orbit.



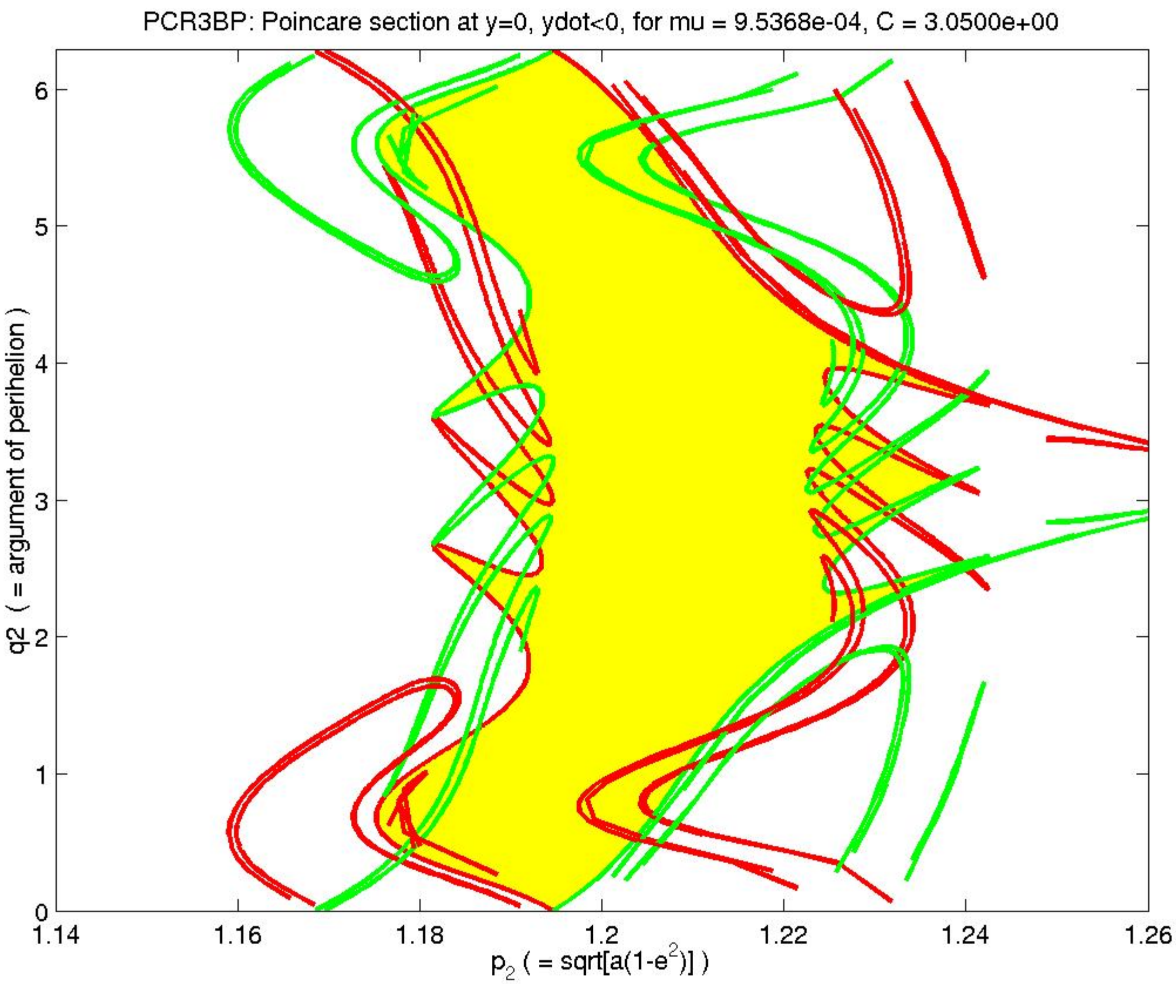
Homoclinic tangle

- Unstable/stable manifolds of periodic points understood as the backbone of the dynamics. This is the homoclinic tangle glimpsed by Poincaré.



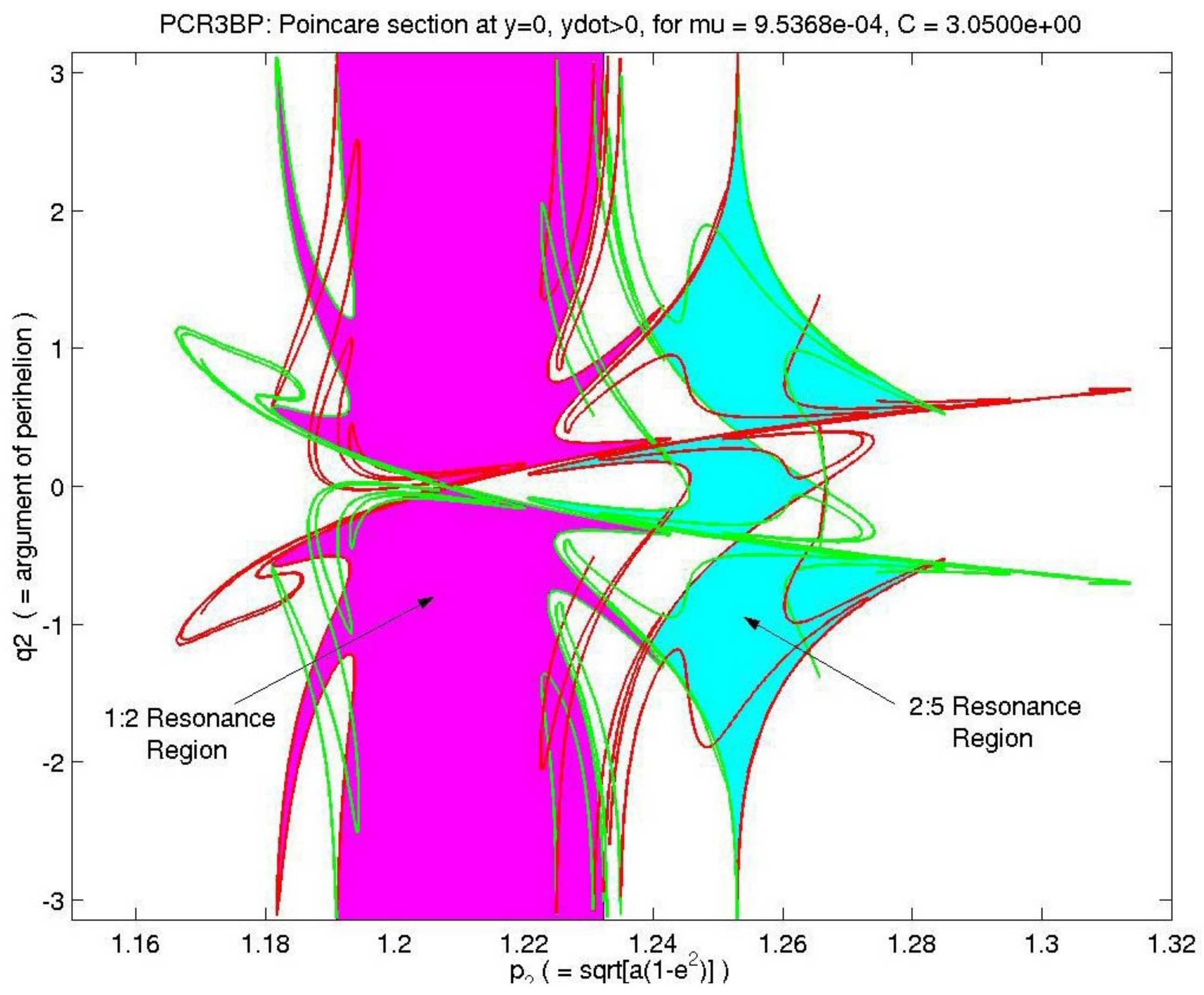
Resonance region

- Unstable/stable manifolds up to “pip” (cf. Wigging [1992]) denote the boundary of the **resonance region.**



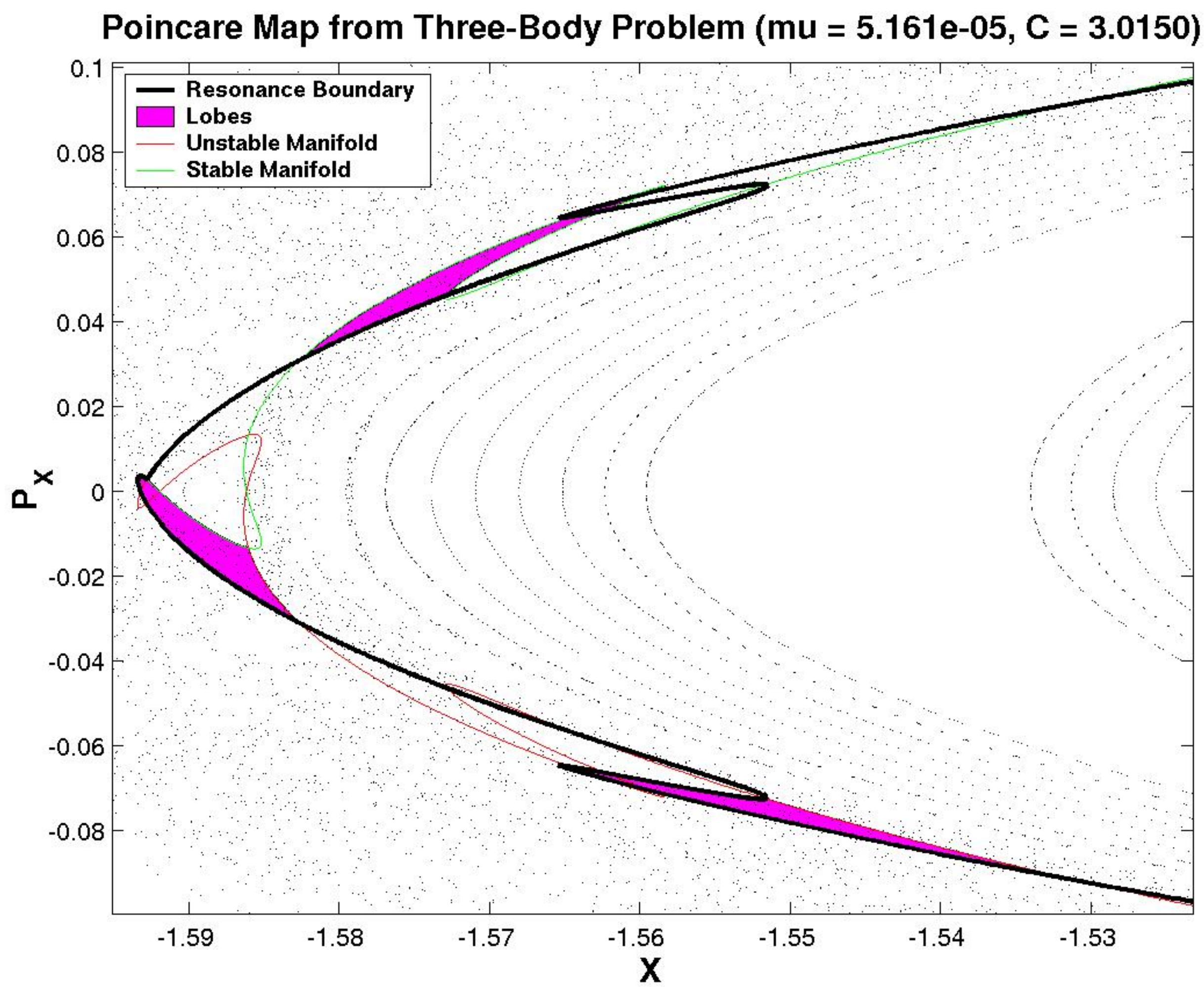
Resonance region

- Neighboring resonance regions indeed **overlap**, leading to complicated mixing.



Transport quantities

- **Lobe dynamics**; following intersections of stable and unstable invariant manifolds of periodic orbits (Wiggins et al.)



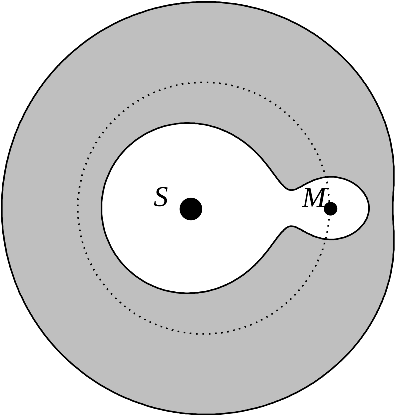
Transport quantities

- These methods are preferred over the **“brute force”** solar system calculations seen in the literature since they are based on first principles.
- Reveal generic structures; give deeper insight.

Escape rates

Obtain rates and probabilities

- One can compute the rate of escape of asteroids temporarily captured by Mars.
  - Jaffé, Ross, Lo, Marsden, Farrelly, and Uzer [2002]
- Statistical approach
  - similar to chemical dynamics, see Truhlar [1996]
- Consider an asteroid (or other body) in orbit around Mars (perhaps impact ejecta) at a 3-body energy such that it can escape toward the Sun.
- Interested in rate of escape of such bodies at a fixed energy, i.e. $F_{M,S}(t)$

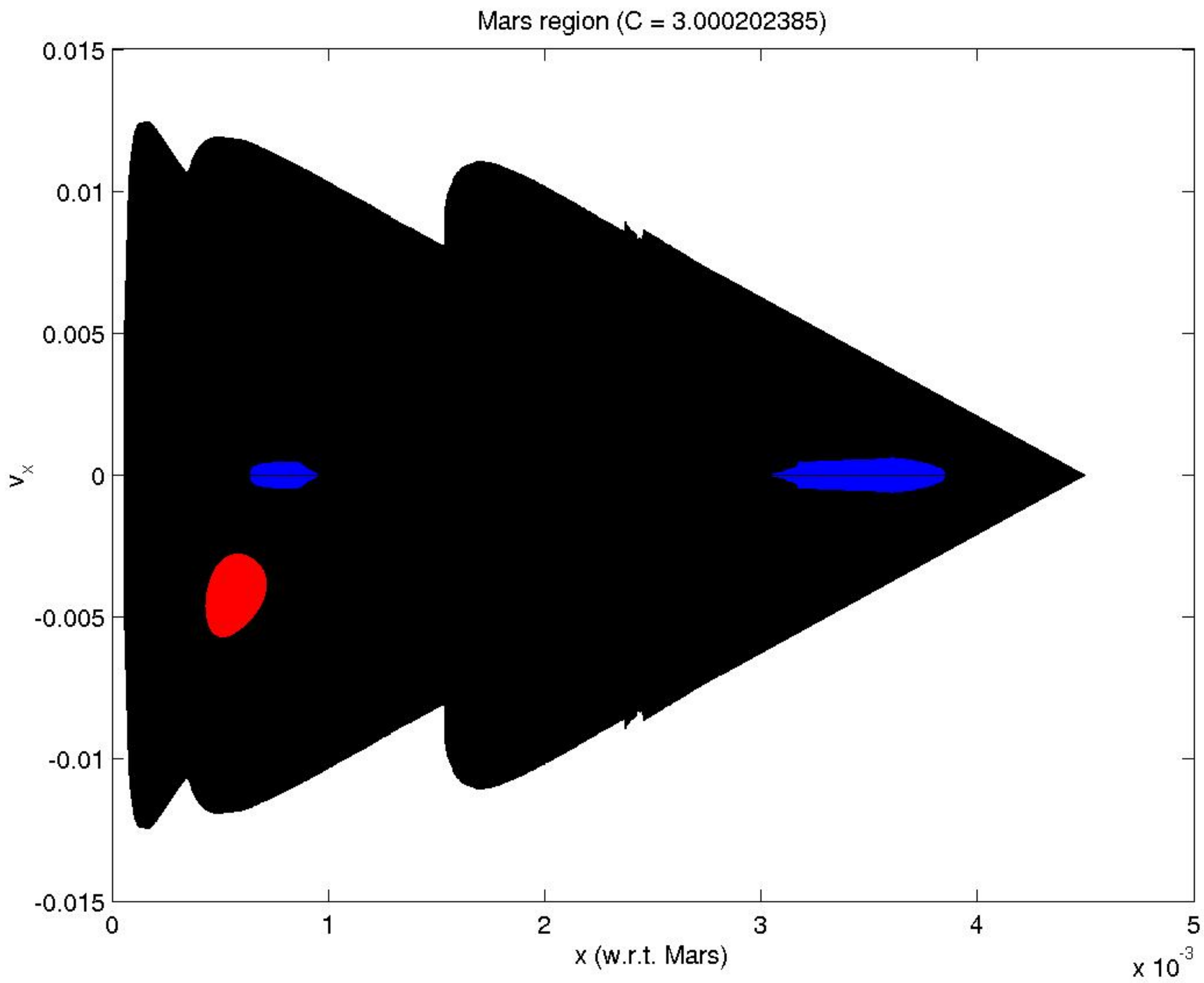

S
M

Escape rates

- Mixing assumption: all asteroids in the Mars region at fixed energy are **equally likely to escape**.
-

$$\begin{aligned}\text{Escape rate} &= \frac{\text{flux out of Mars region}}{\text{Mars region phase space volume}} \\ &= \frac{\text{area of escaping orbits}}{\text{area of chaotic region}}\end{aligned}$$

Escape rates

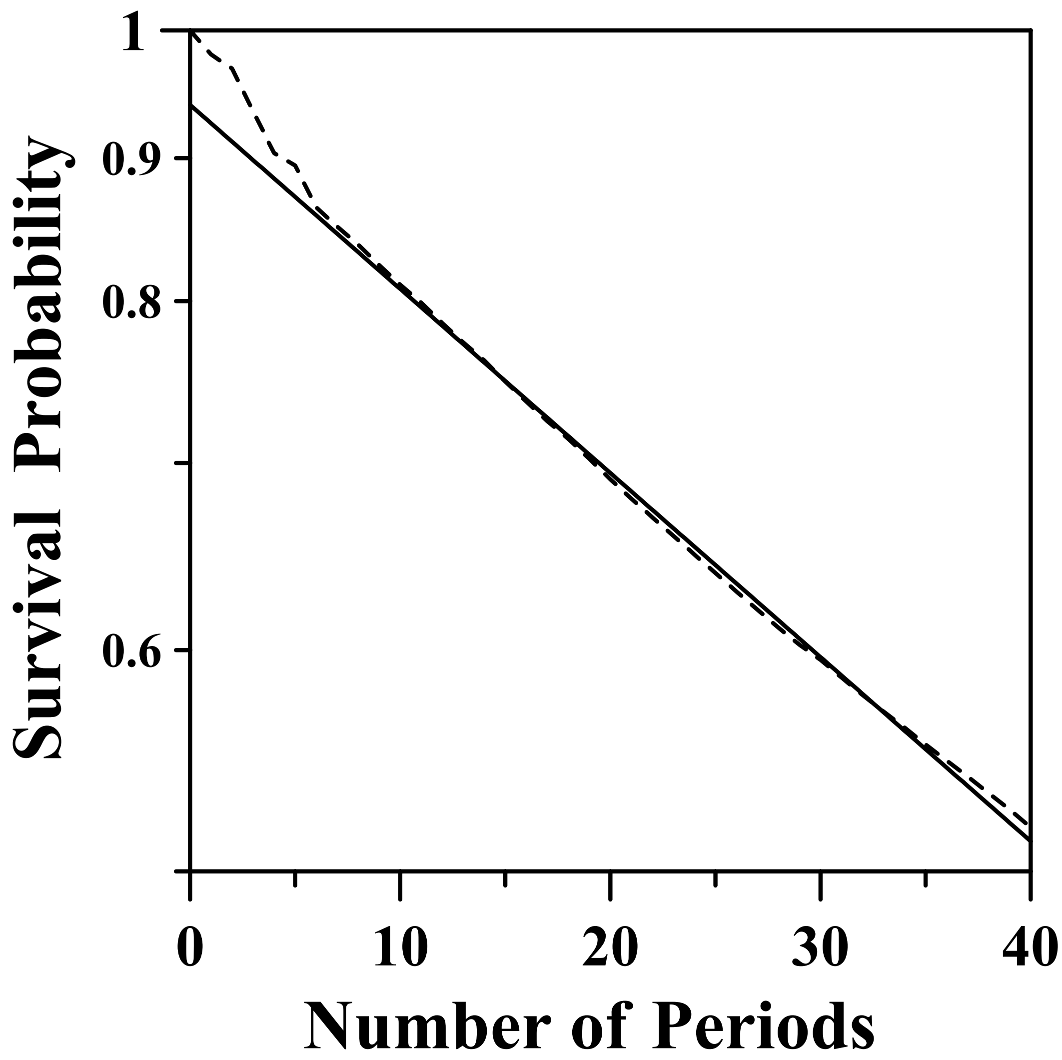


Escape rates

- Compare with Monte Carlo simulations of 107,000 particles
  - randomly selected initial conditions at constant energy

Transition Rates

- Theory and numerical simulations agree well.
  - Monte Carlo simulation (dashed) and theory (solid)



Steady state distribution

- If the planar, circular restricted three-body problem is **ergodic,** then a statistical mechanics can be built (cf. ZhiGang [1999]).
- Recent work suggests there may be regions of the energy shell for which the motion is ergodic, in particular the "chaotic sea" (Jaffé et al. [2002]).
- This suggests we compute the **steady state distribution** of some observable for particles in the chaotic sea; a simple method for obtaining the likely locations of any particles within it.

Steady state distribution

- Assuming ergodicity,

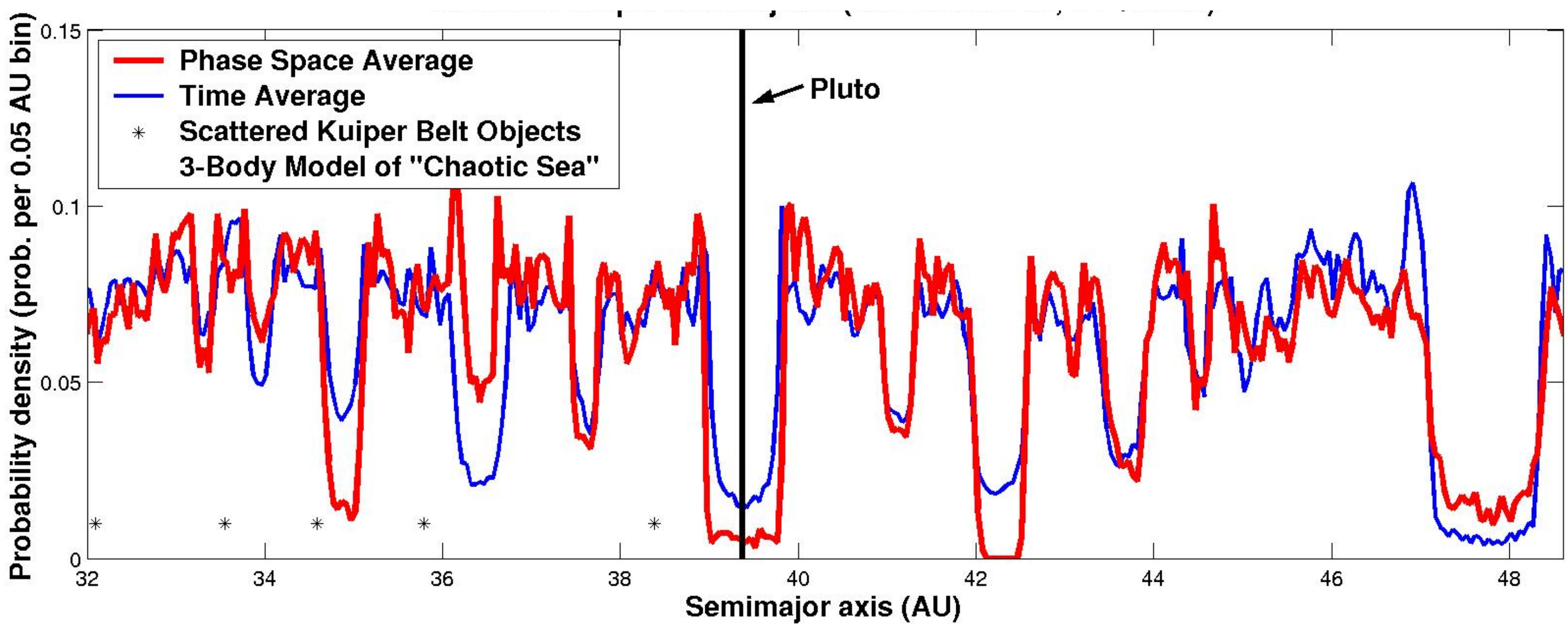
$$\lim_{t\to\infty} \frac{1}{t} \int_0^t A(x, y, p_x, p_y) d\tau = \int A(x, y, p_x, p_y) \frac{C}{|\frac{\partial H}{\partial p_y}|} dp_x dx dy,$$

where $A(x, y, p_x, p_y)$ is any physical observable (e.g., semimajor axis), one can finds that the density function, $\rho(x, p_x)$, on the surface-of-section, $\Sigma_{(\mu,\varepsilon)}$, is constant.

- We can determine the steady state distribution of semi-major axes; define $N(a)da$ as the number of particles falling into $a \longrightarrow a + da$ on the surface-of-section, $\Sigma_{(\mu,\varepsilon)}$.

Steady state distribution

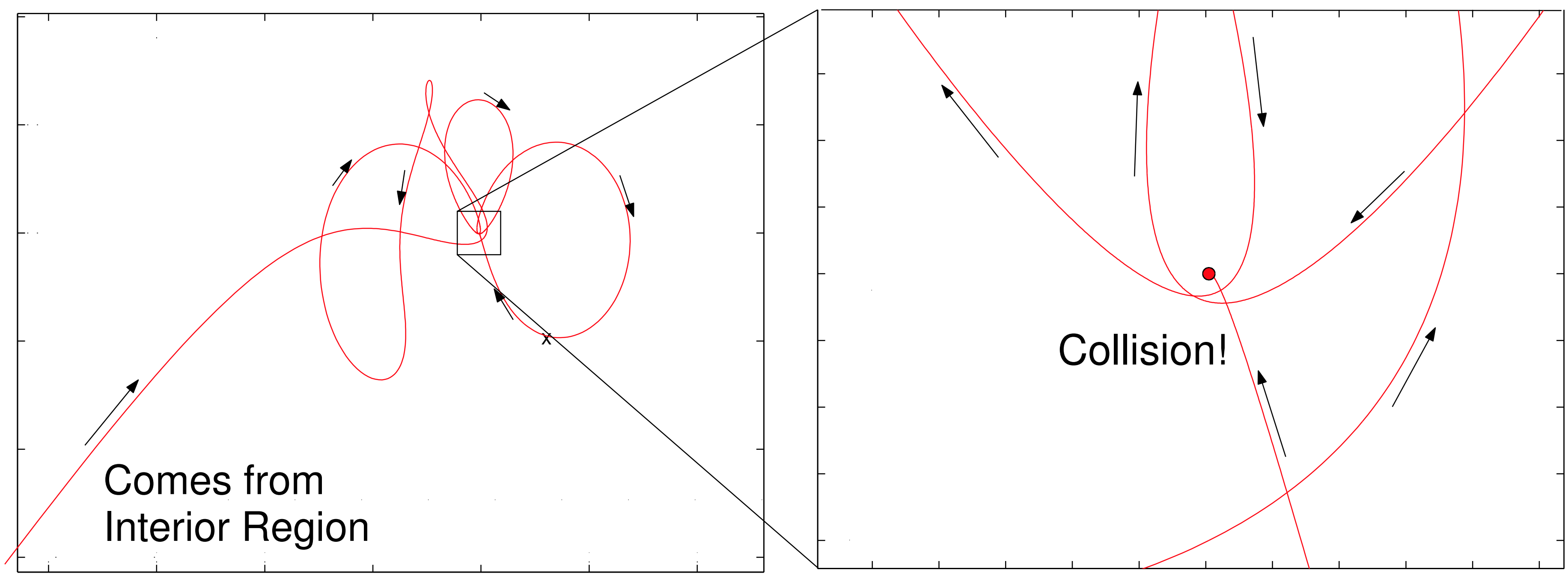
- SKBOs should be in regions of high density.



Collision Probabilities

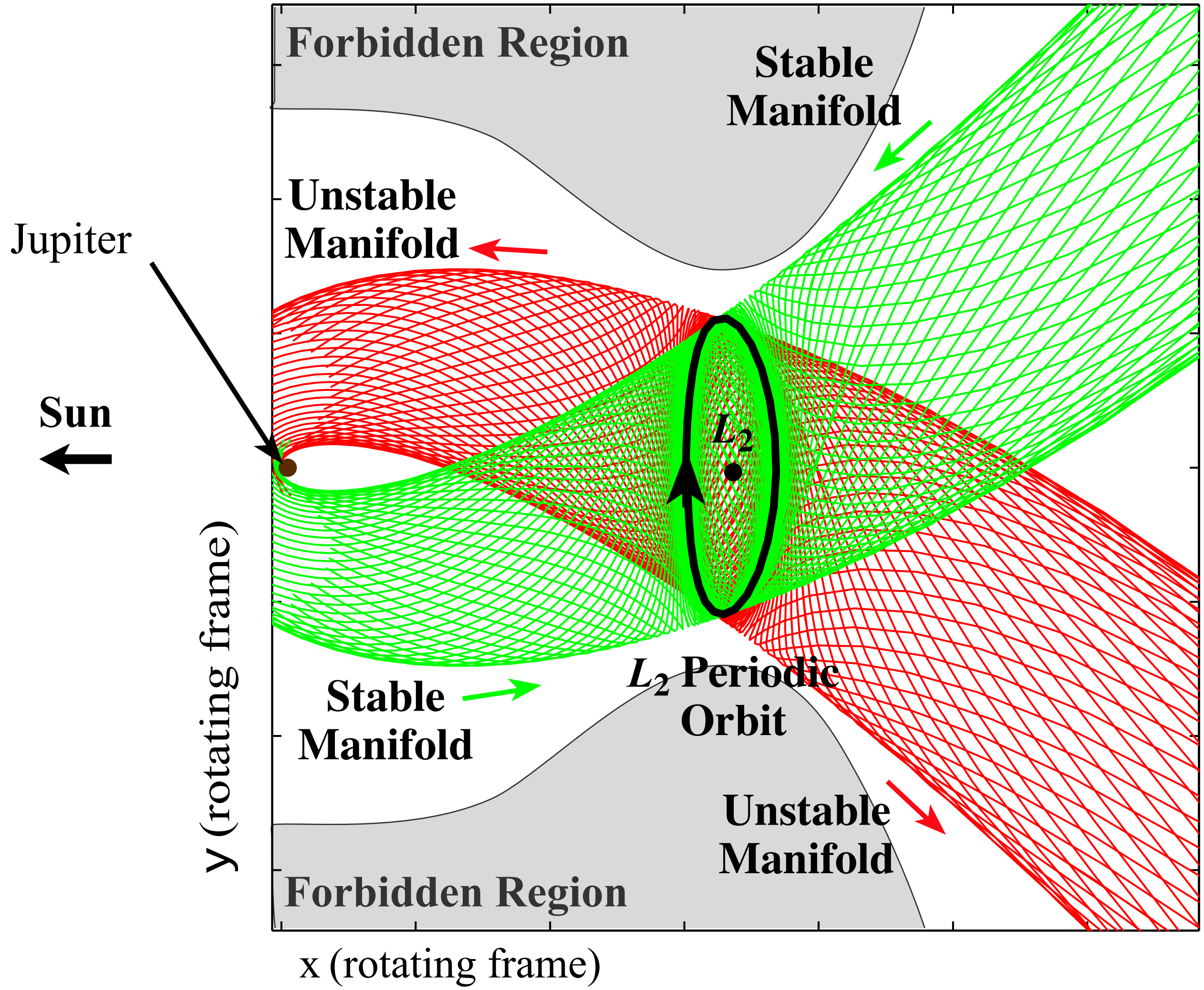
- Low velocity impact probabilities
- Assume object enters the planetary region with an energy slightly above L1 or L2
  - eg, **Shoemaker-Levy 9** and **Earth-impacting asteroids**

Example Collision Trajectory



Tubes in the 3-Body Problem

- **Stable** and **unstable manifold tubes**
  - Control transport through the neck.



Collision Probabilities

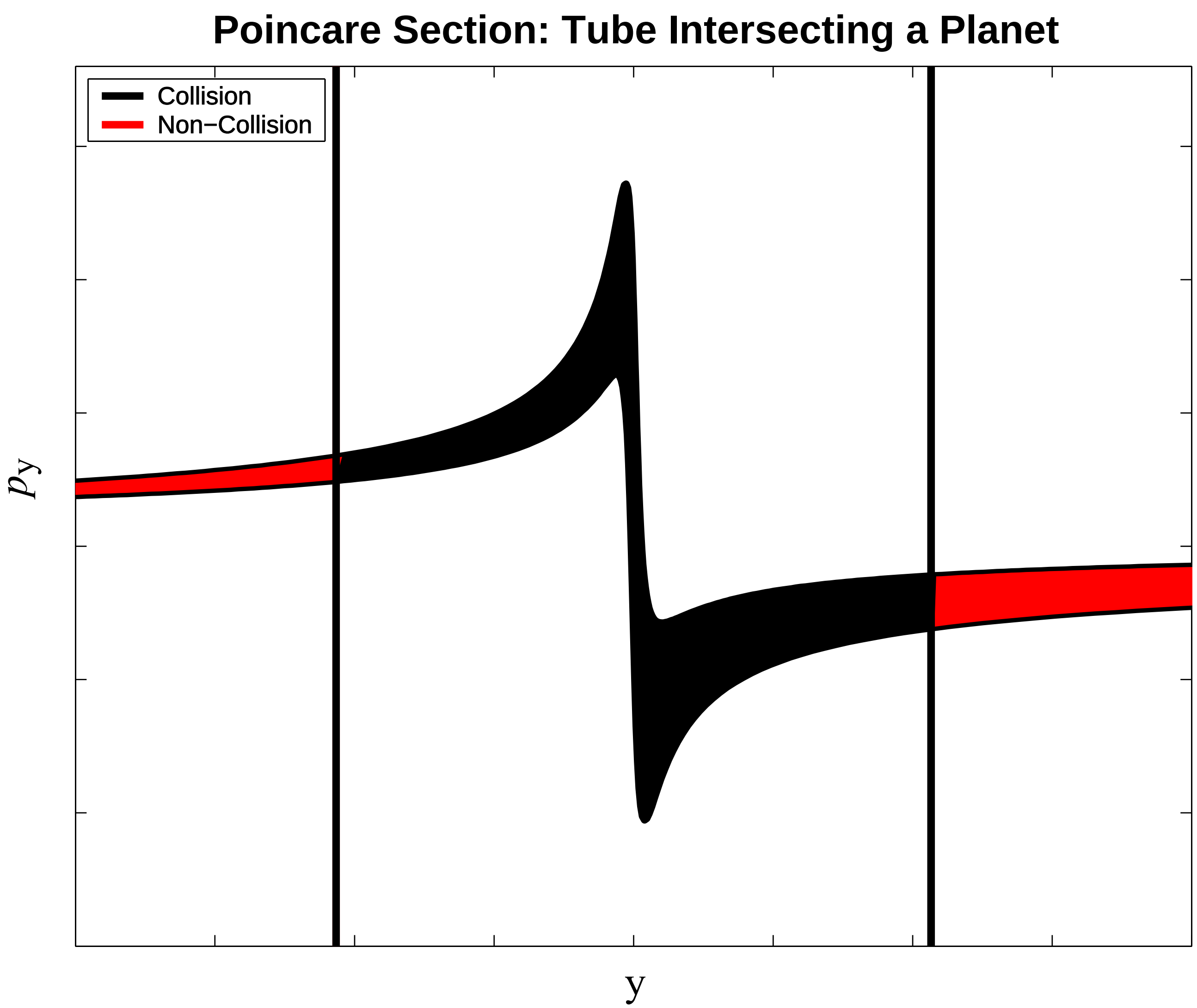
Collision probabilities

- Compute from tube intersection with planet on Poincaré section
- Planetary diameter is a parameter, in addition to μ and energy E

$\leftarrow$ Diameter of planet $\rightarrow$

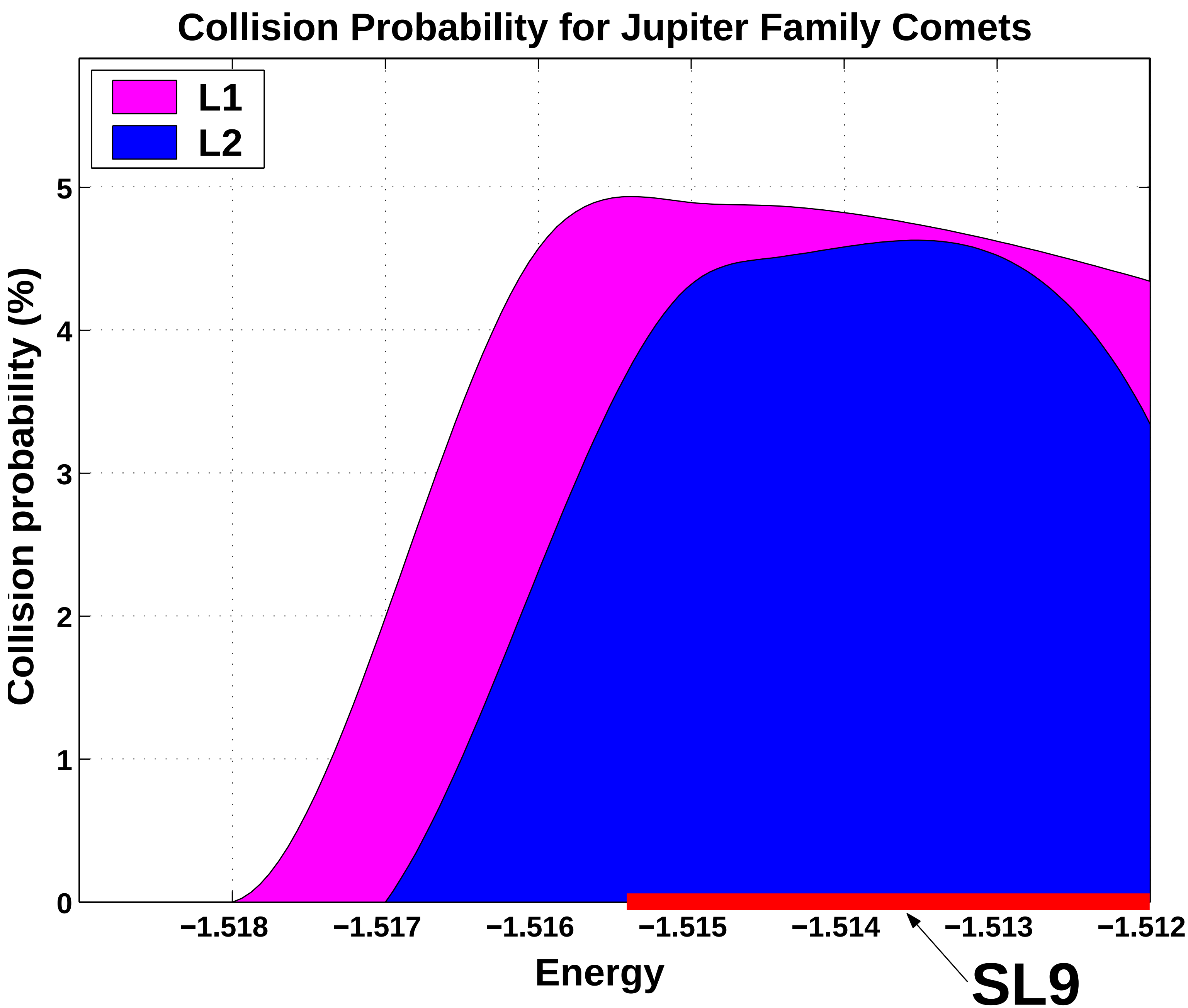
Collision Probabilities

Collision probabilities

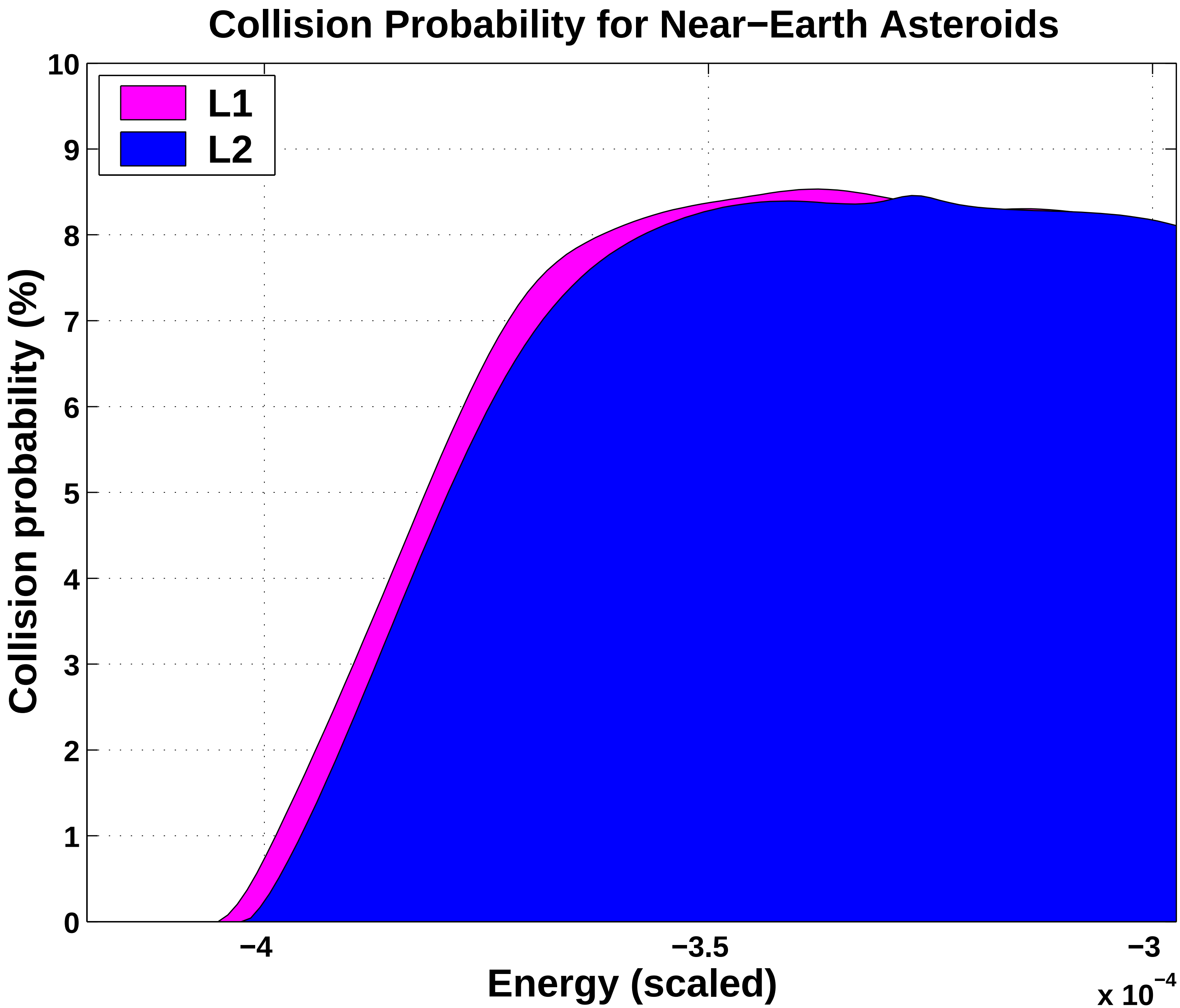


$\longleftarrow$ Diameter of planet $\longrightarrow$

Collision Probabilities



Collision Probabilities



Conclusion and Future Work

- ***Transport in the solar system***
  - Approximate some solar system phenomena using the restricted 3-body problem
  - Circular restricted 3-body problem
    - Stable and unstable manifold tubes of libration point orbits can be used to compute statistical quantities of interest
    - Probabilities of transition, collision
  - Theory and observation agree
- ***Future studies to involve multiple three-body problems and 3-d.o.f.***

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