CALTECH
Control & Dynamical Systems

Phase space transport. II

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Full Body Problem Workshop

Caltech, November 14-15, 2003

Motivation

- Apply transport calculations to asteroid pairs to calculate, e.g., capture & escape rates.

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Dactyl in orbit about Ida, discovered in 1994 during the *Galileo* mission.

Motivation

- Movie of simple model of Ida & Dactyl
- Small hard sphere around rotating elliptical asteroid
- Nonmerging collisions modeled as bounces
  (work with E. Kanso)

In This Talk...

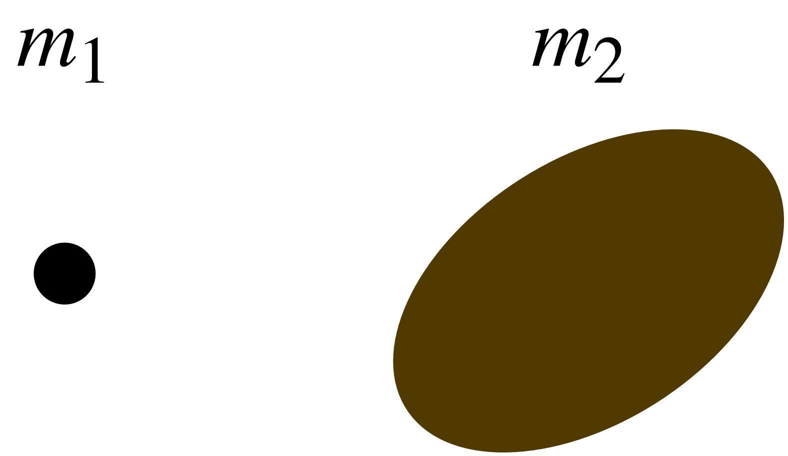
- ***Part I. Restricted F2BP phase space***
  - Dependence on energy
  - Tube + lobe dynamics
  - Modeling collisions
- ***Part II. Transport using set oriented methods***
  - Transport problem described
  - Two computational techniques
    (a) Invariant manifolds
    (b) Almost-invariant sets
  - Extensions and future work

Part I

Restricted Full Two Body Problem

- Consider two masses: m_1 (sphere) & m_2 (ellipse)
- $\frac{m_1}{m_2} \longrightarrow 0$
  Particle around asteroid

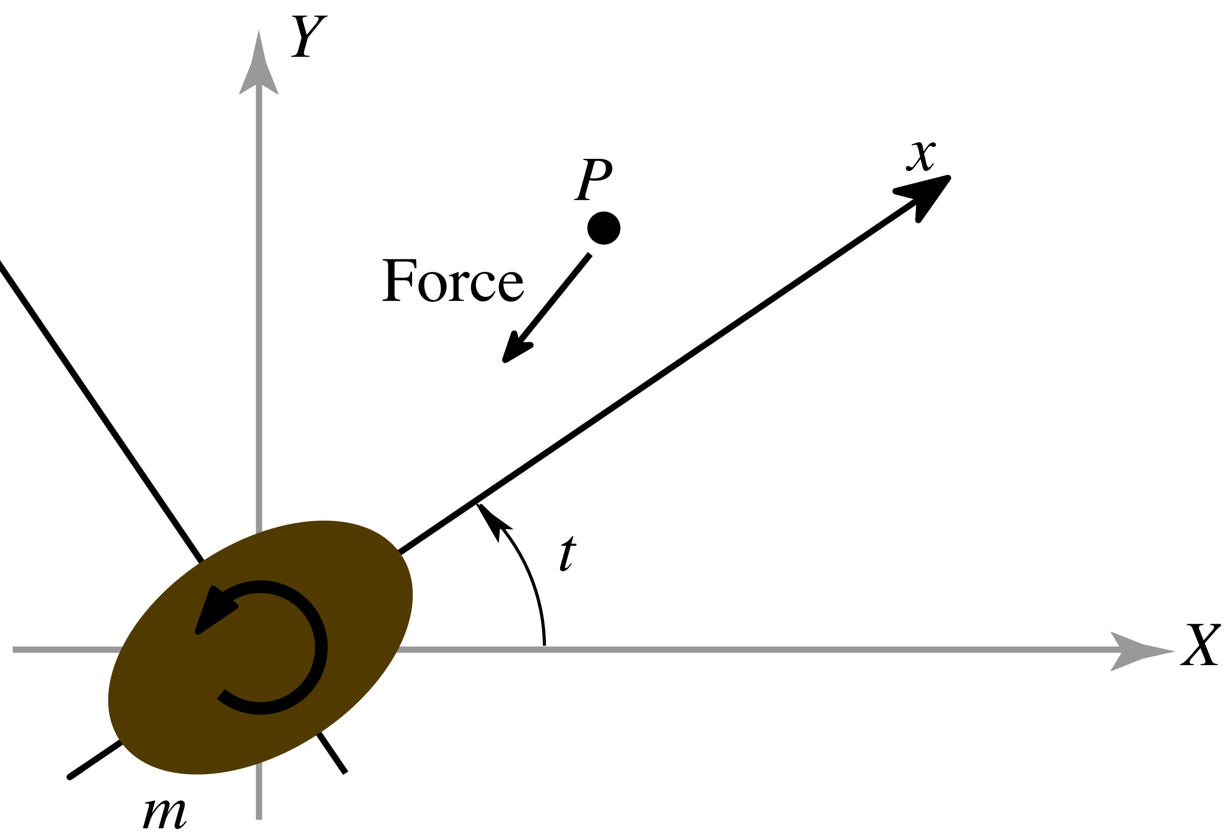
  restricted F2BP (**RF2BP**)



- Restricted (as in restricted 3-body problem) simple case exhibits the basic capture, ejection, collision dynamics (see Koon, Marsden, Ross, Lo, Scheeres [2003])
- Can include bouncing, sticking, ... (see Kanso [2003])

Restricted F2BP

- Point mass P moving in the x-y plane under the gravitational field of a uniformly rotating elliptical body m, without affecting its uniform rotation.



The rotating (x-y) and inertial (X-Y) frames.

Restricted F2BP

- **Equations of motion** relative to a rotating Cartesian coordinate frame and appropriately normalized:

$$\ddot{x} - 2\dot{y} = -\frac{\partial U}{\partial x} \quad \text{and} \quad \ddot{y} + 2\dot{x} = -\frac{\partial U}{\partial y},$$

where

$$U(x, y) = -\frac{1}{r} - \frac{1}{2}r^2 - \frac{3C_{22}\left(x^2 - y^2\right)}{r^5},$$

and

$$r = \sqrt{x^2 + y^2}.$$

- **Energy integral** (Jacobi integral):

$$E = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + U(x, y).$$

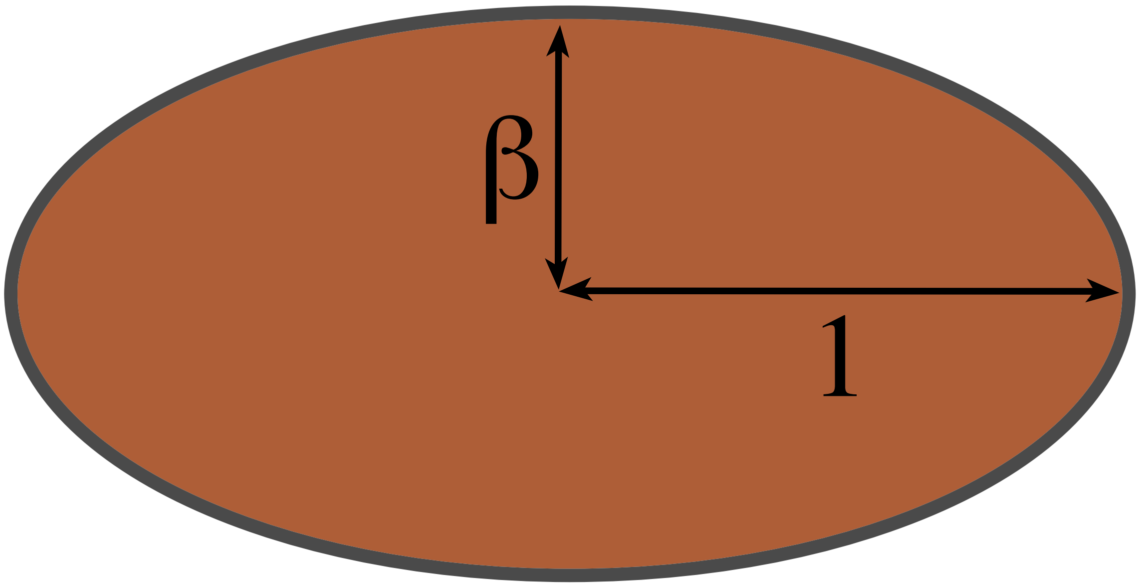
Restricted F2BP

- Gravity field coefficient C_{22}, the **ellipticity**,

$$C_{22} = \frac{1}{20}(1 - \beta^2),$$

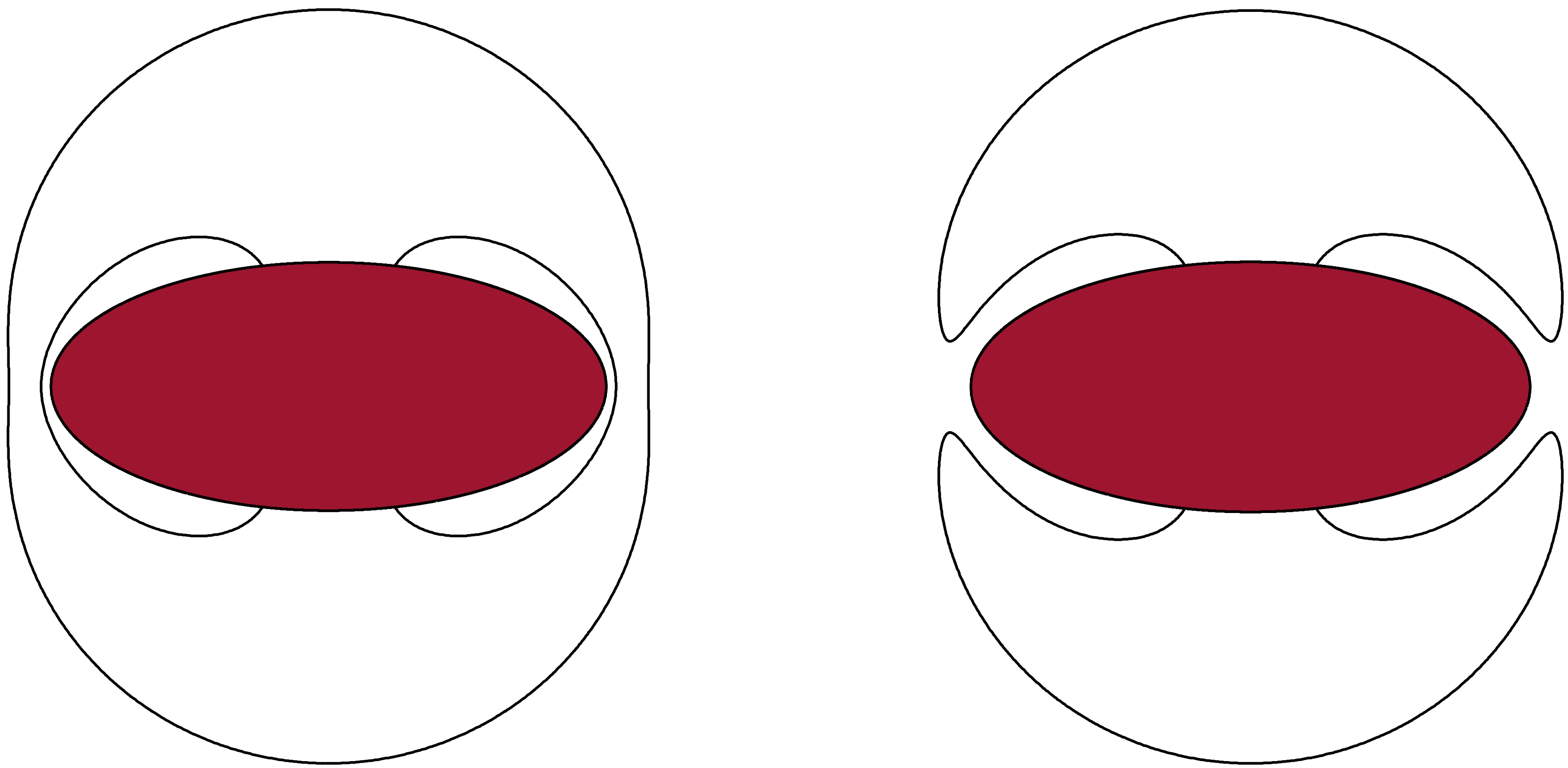
  varies between 0 and 0.05.

- e.g., Ida: $\beta \approx 0.43, C_{22} \approx 0.04$



Phase Space Structure

- Energy indicates type of global dynamics.
- Is movement between the **exterior** and **asteroid realms** possible?



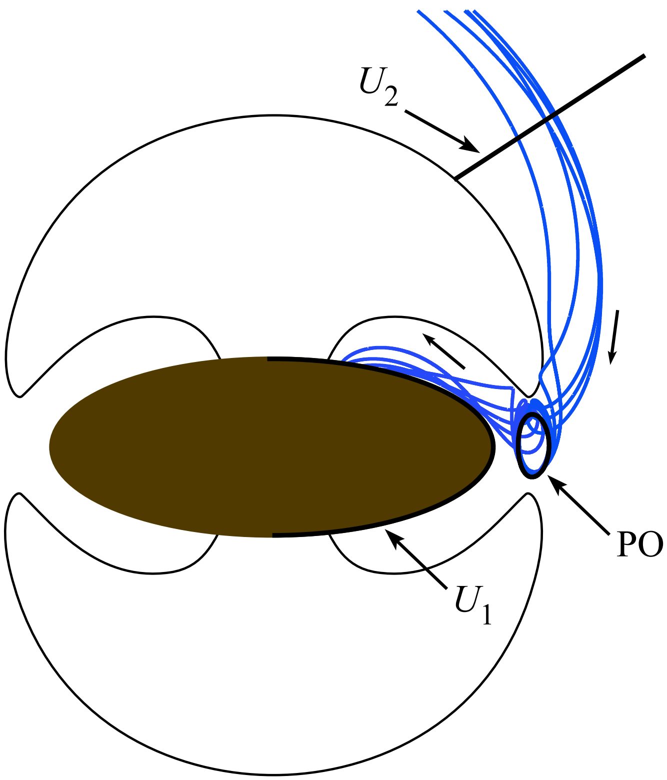
$E < E_S$ $\qquad$ $E > E_S$

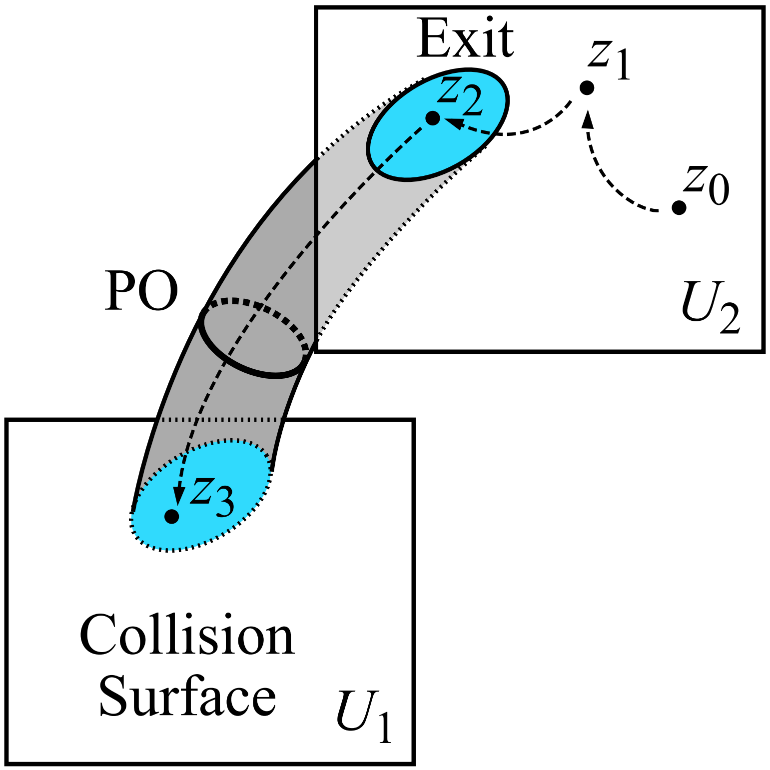
Phase Space Structure

- *Multi-scale dynamics*
  - Coarse level : **tube dynamics** between realms
  - Fine level : **lobe dynamics** within realms

Phase Space Structure

- Slices of energy surface: Poincaré sections U_i
- Tube dynamics: evolution **between** U_i
- Lobe dynamics: evolution **on** U_i





Lobe Dynamics

- Suppose $E < E_S$; energy surface $\mathcal{M}_E$
- Asteroid and exterior realms not connected
- Poincaré map in exterior realm: area and orientation preserving map on $M \subset \mathbb{R}^2$,

$$f : M \longrightarrow M$$

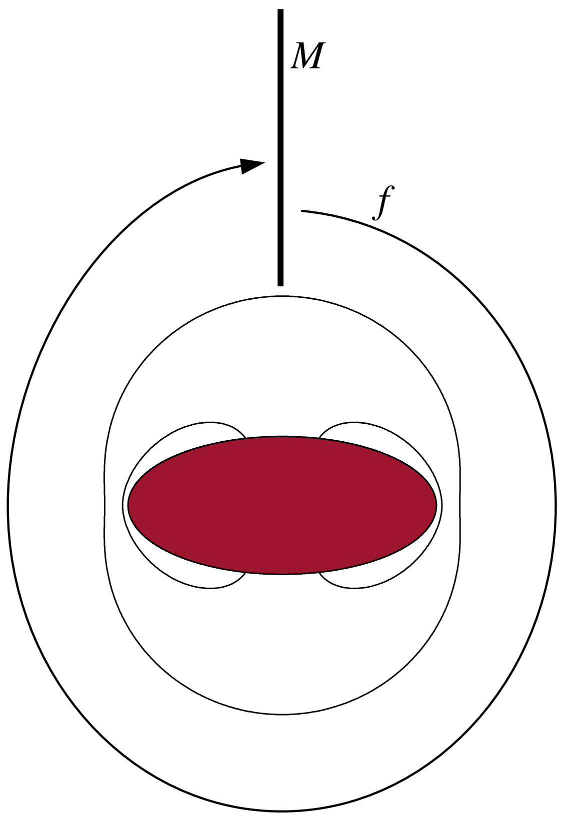
where

$$M = \mathcal{M}_E \cap \{x = 0, \dot{x} > 0\}$$

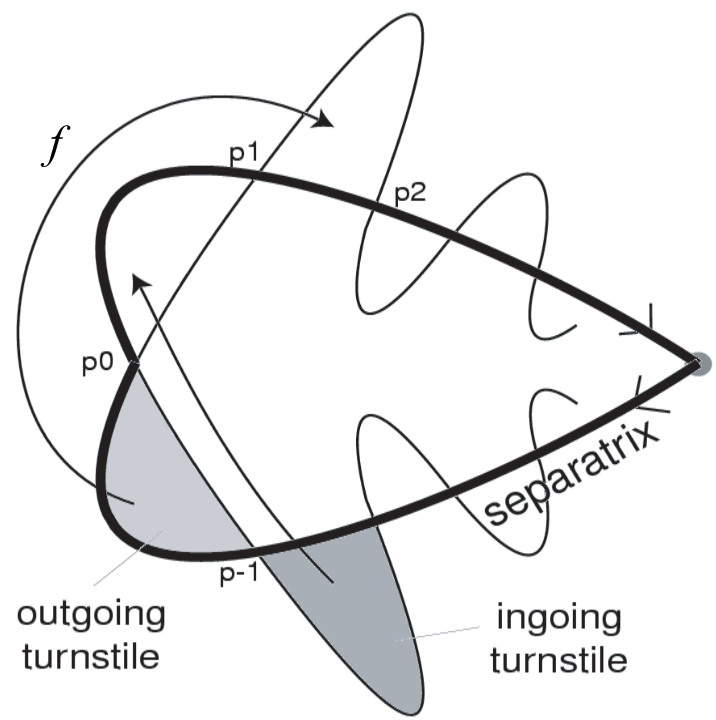
with coordinates (y, p_y), equiv., (r, p_r), on M

Lobe Dynamics

- Particles are **ejected** if they lie within lobes enclosed by the stable and unstable manifolds of a hyperbolic fixed point at $(+\infty, 0)$—**lobes of ejection**.

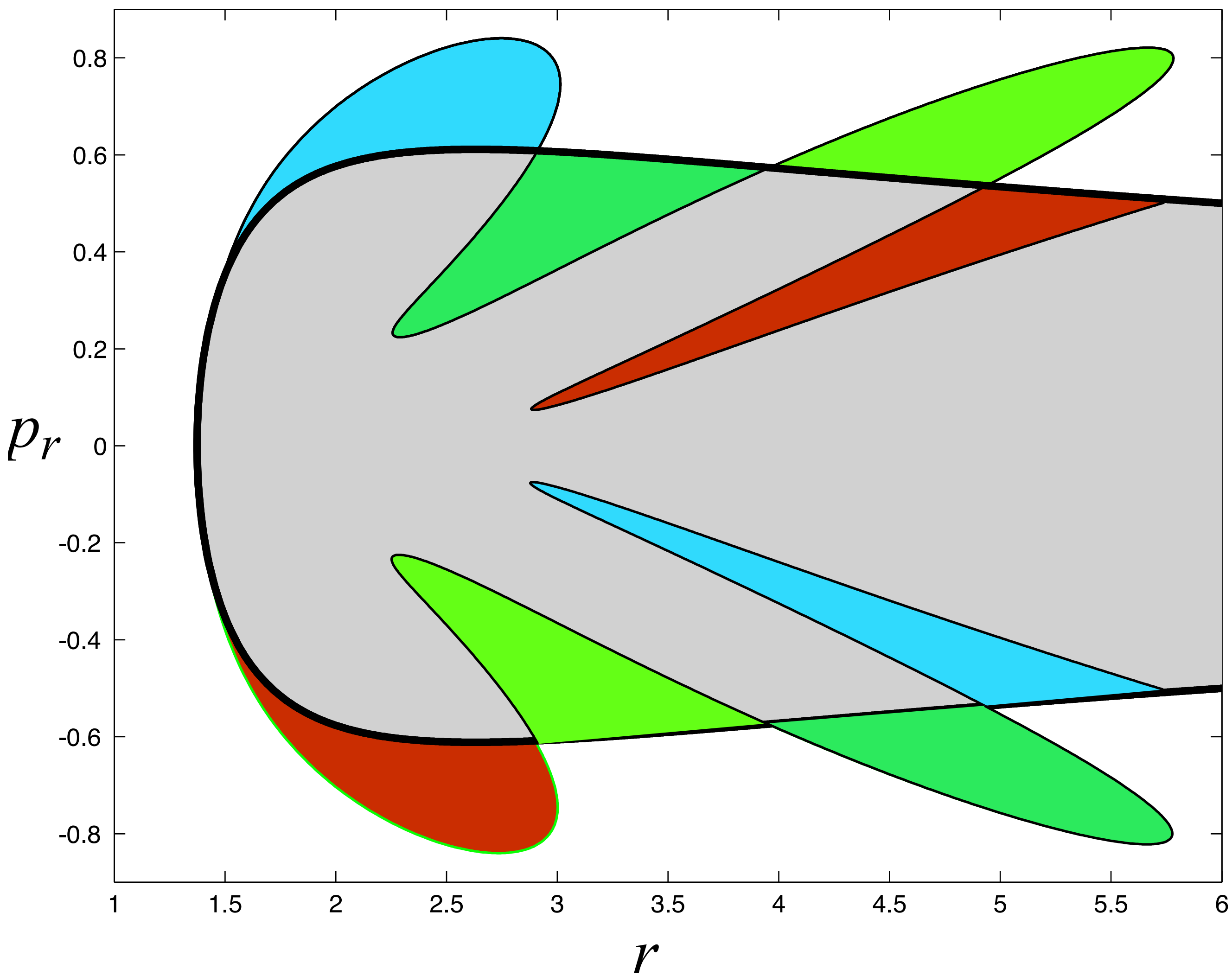


Position space projection



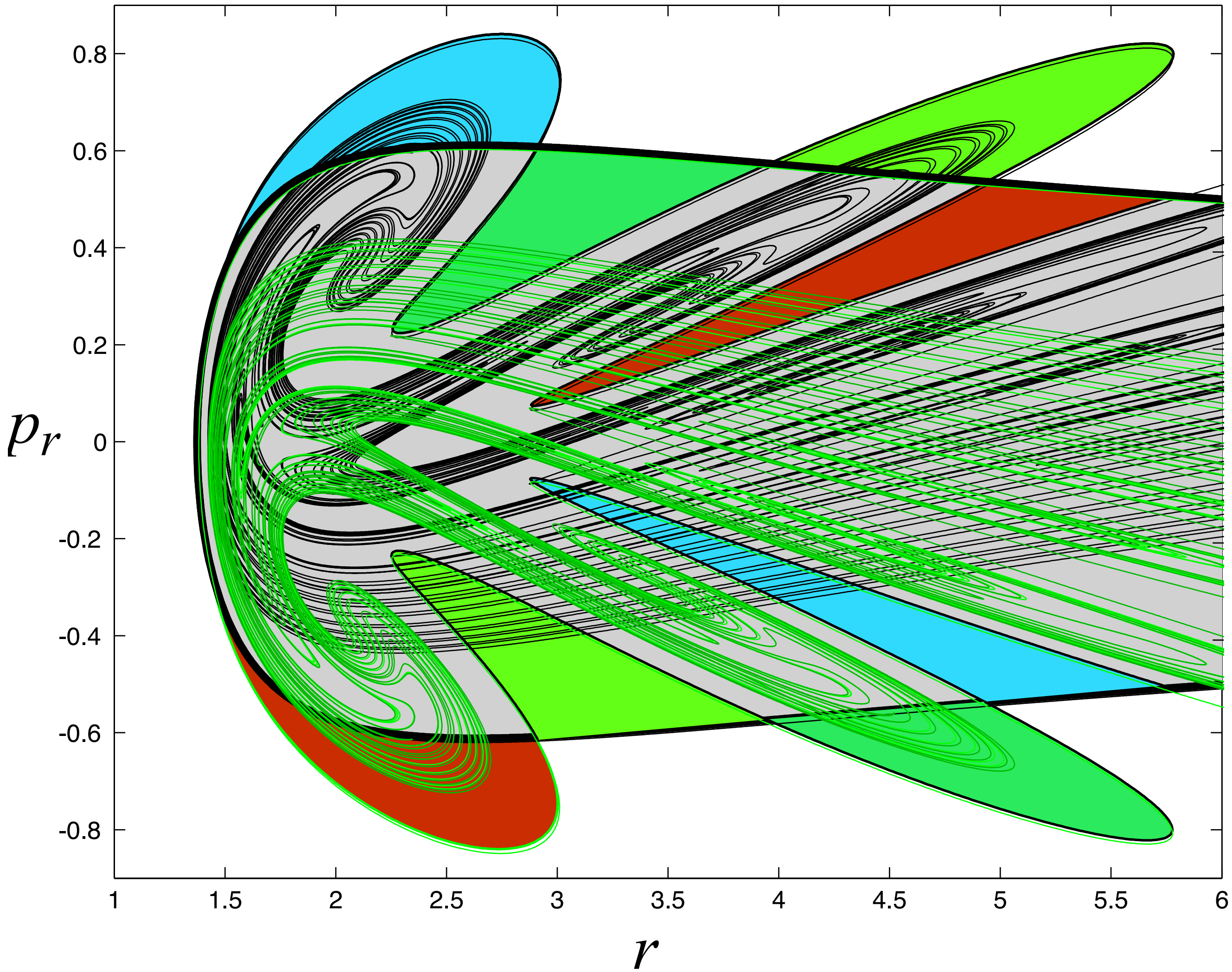
Motion on M

Lobe Dynamics



Numerical simulation using MANGEN

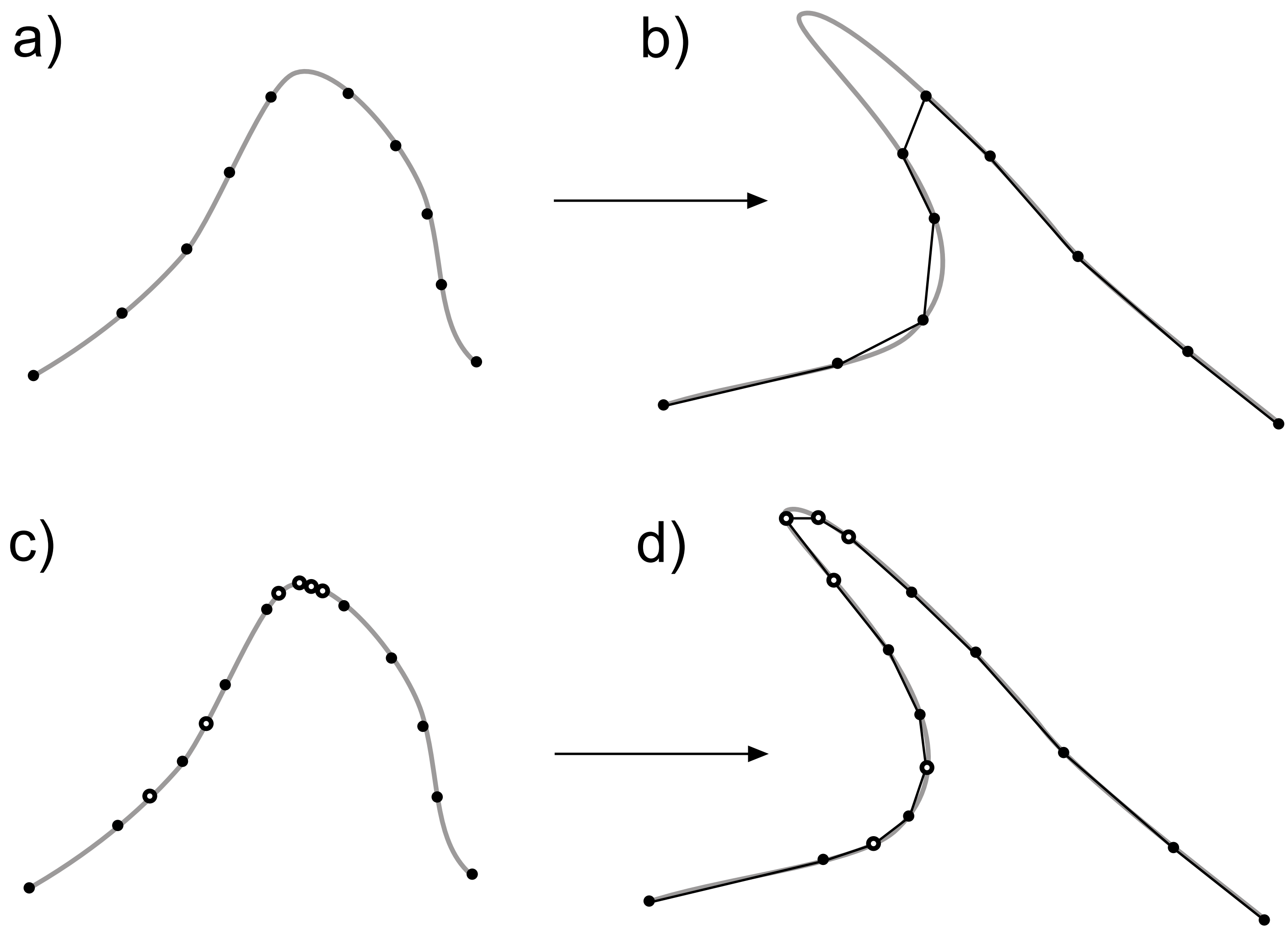
Lobe Dynamics



Curves can be followed to very high accuracy

MANGEN Description

- Simulations use MANGEN (Coulliete & Lekien)
- Adaptive conditioning of curves based on curvature.

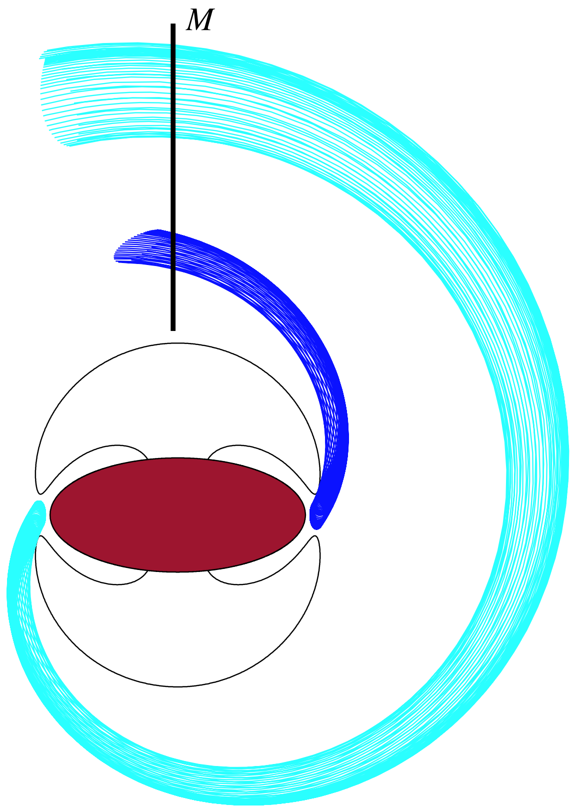


Tube + Lobe Dynamics

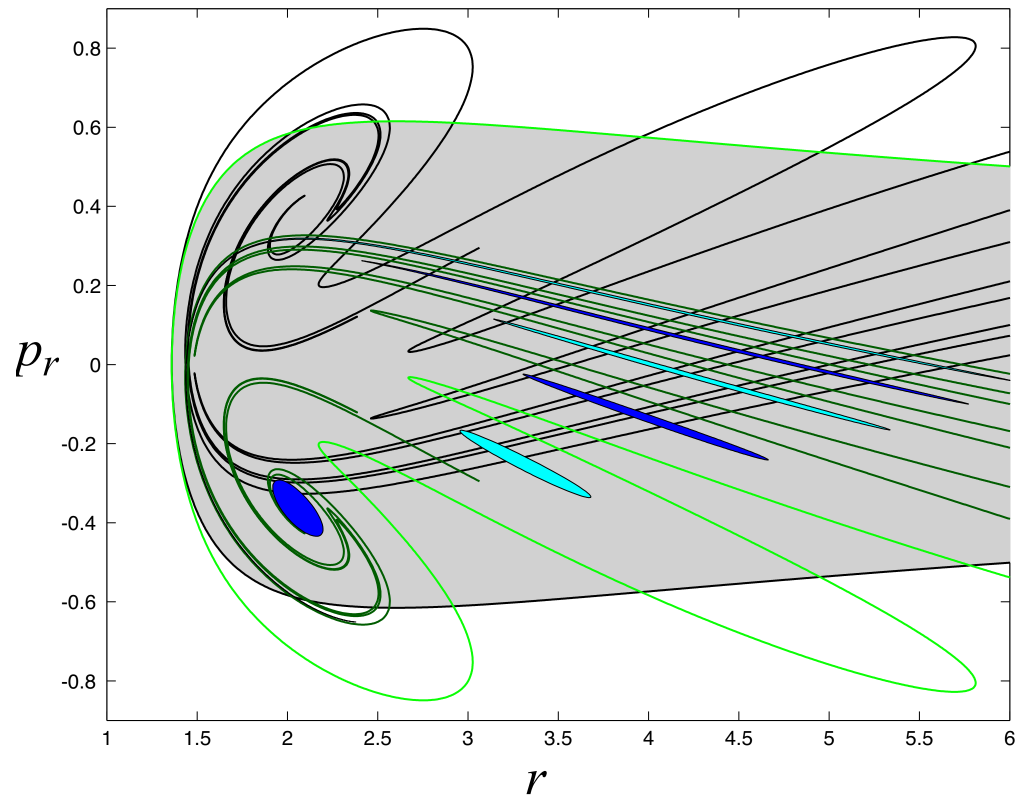
- Suppose $E > E_S$
- Exterior and asteroid realms **connected via tubes**
- In exterior realm, some tubes lead to collision (others lead away from collision)
- **Tube + lobe dynamics =** Alternate fates of collision and ejection are intimately intermingled.

Tube + Lobe Dynamics

- Tubes leading to collision with asteroid



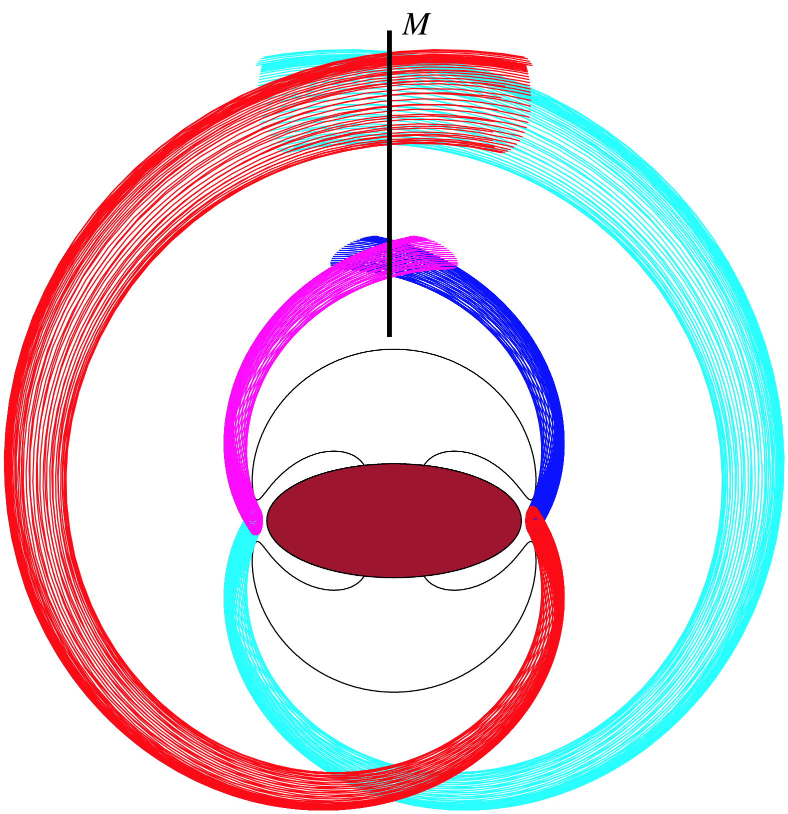
Position space projection



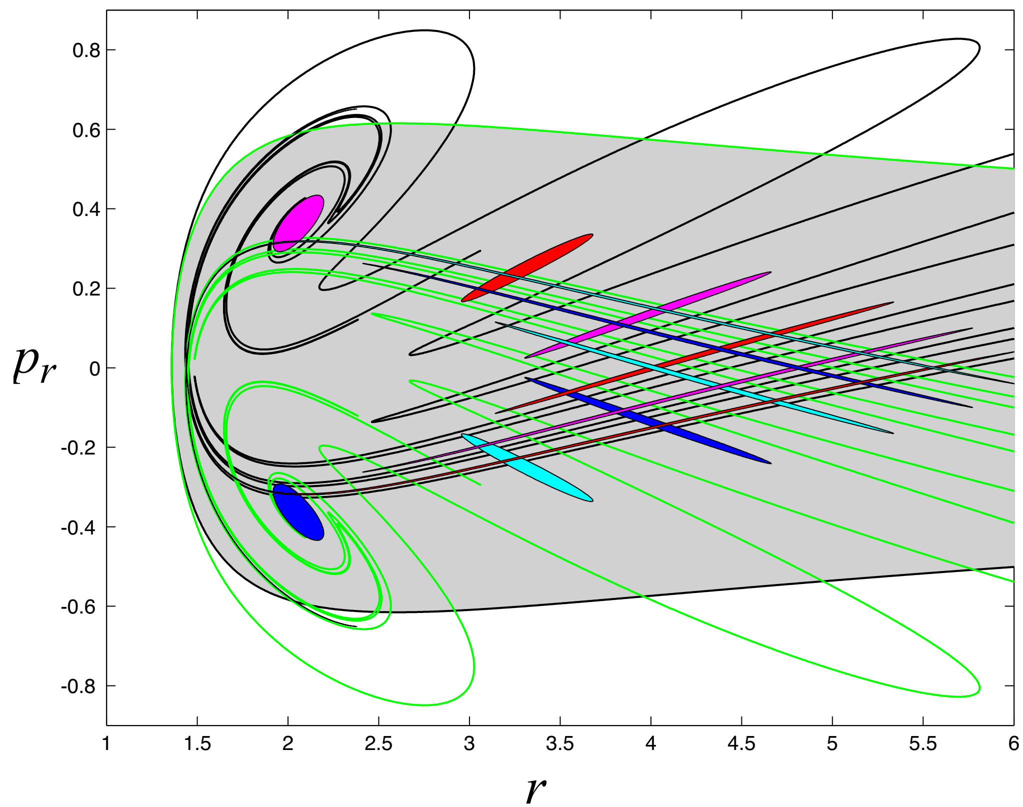
Motion on M

Tube + Lobe Dynamics

- Tubes leading to collision with asteroid
  plus tubes coming from collision, e.g., liberated particles



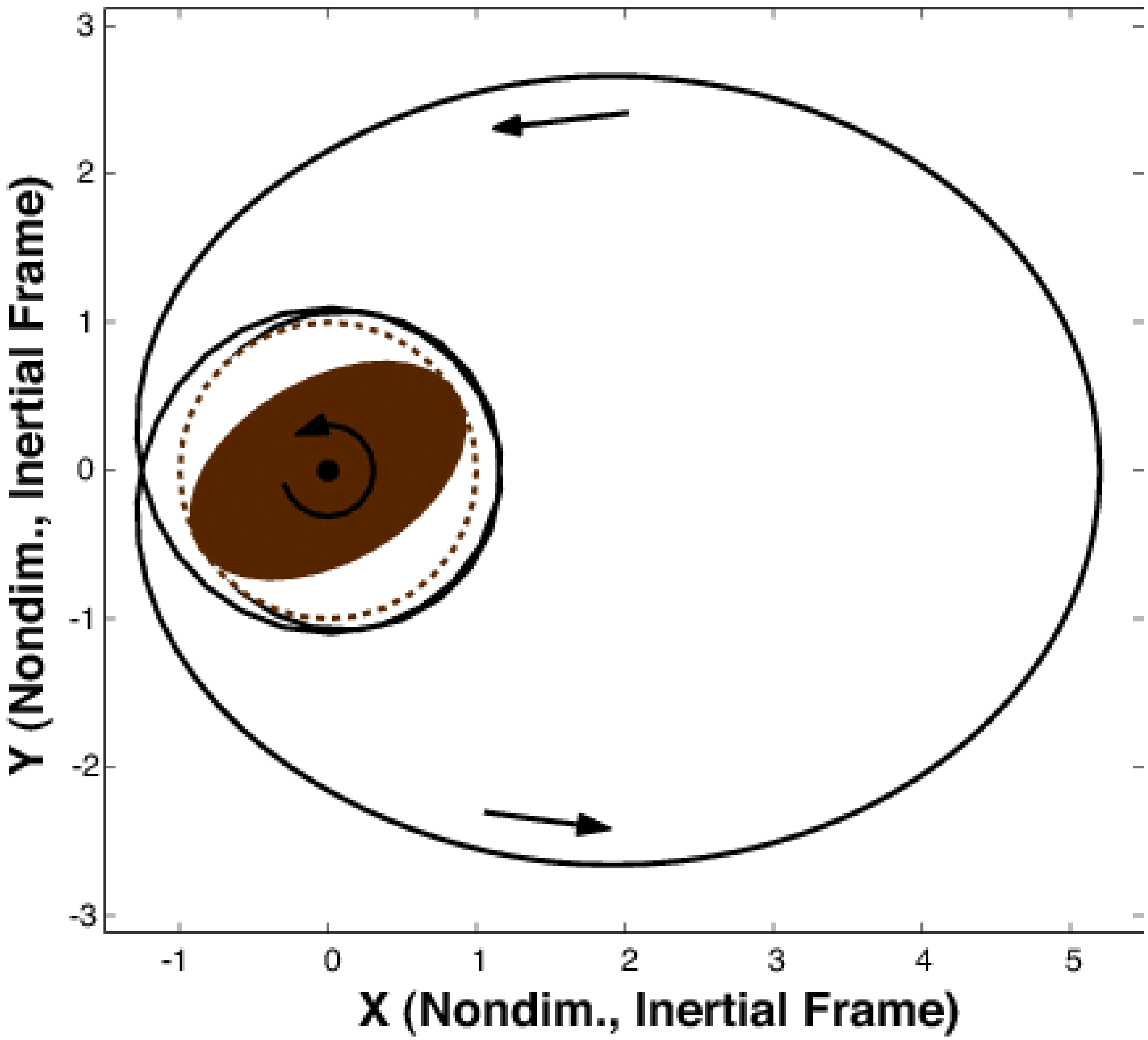
Position space projection



Motion on M

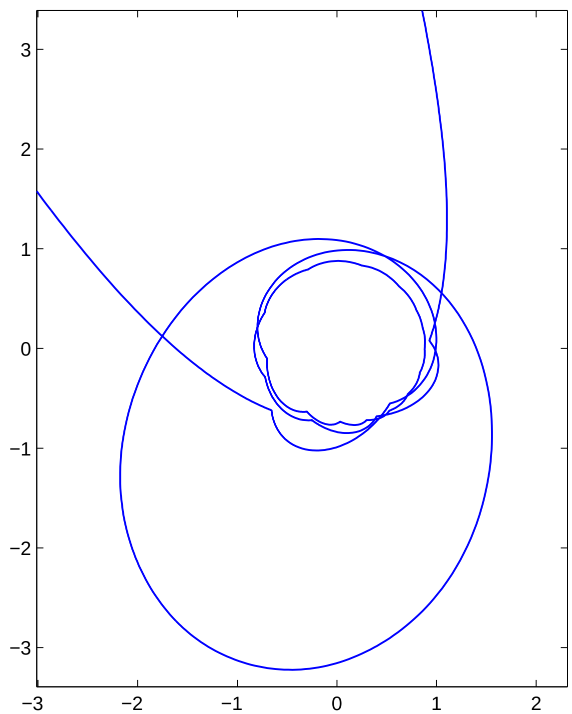
Tube + Lobe Dynamics

- Escape and re-capture.

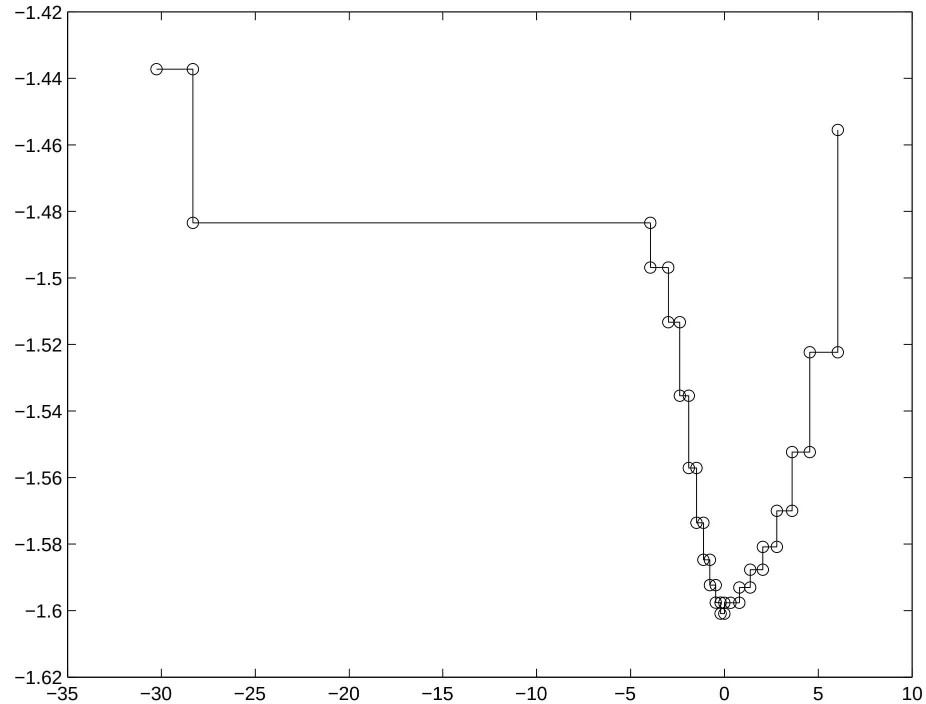


Collision Modeling

- If bouncing is modeled, dynamics is more complicated.
- Upon bouncing, particle moves to new energy surface
- Work in progress with E. Kanso



Inertial frame projection



Energy history

Part II

Transport using set oriented methods

- Describe transport of phase points on a k-dimensional manifold M

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Transport using set oriented methods

- Describe transport of phase points on a k-dimensional manifold M
- M could be, e.g., the ocean surface, an energy shell, or a Poincaré surface-of-section
- We look first at $k = 2$ for autonomous systems
- Paper: Dellnitz, Junge, Koon, Lekien, Lo, Marsden, Padberg, Preis, R., Thiere [2003]

Statement of Problem

- Consider a volume- and orientation-preserving map

$$f : M \to M,$$

on some compact set $M \subset \mathbb{R}^2$ with volume measure μ.
e.g., f may be a discretization of an autonomous flow.

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- Initially, R_i is uniformly covered with **species** S_i. i.e., Species type indicates where a point was initially.

Statement of Problem

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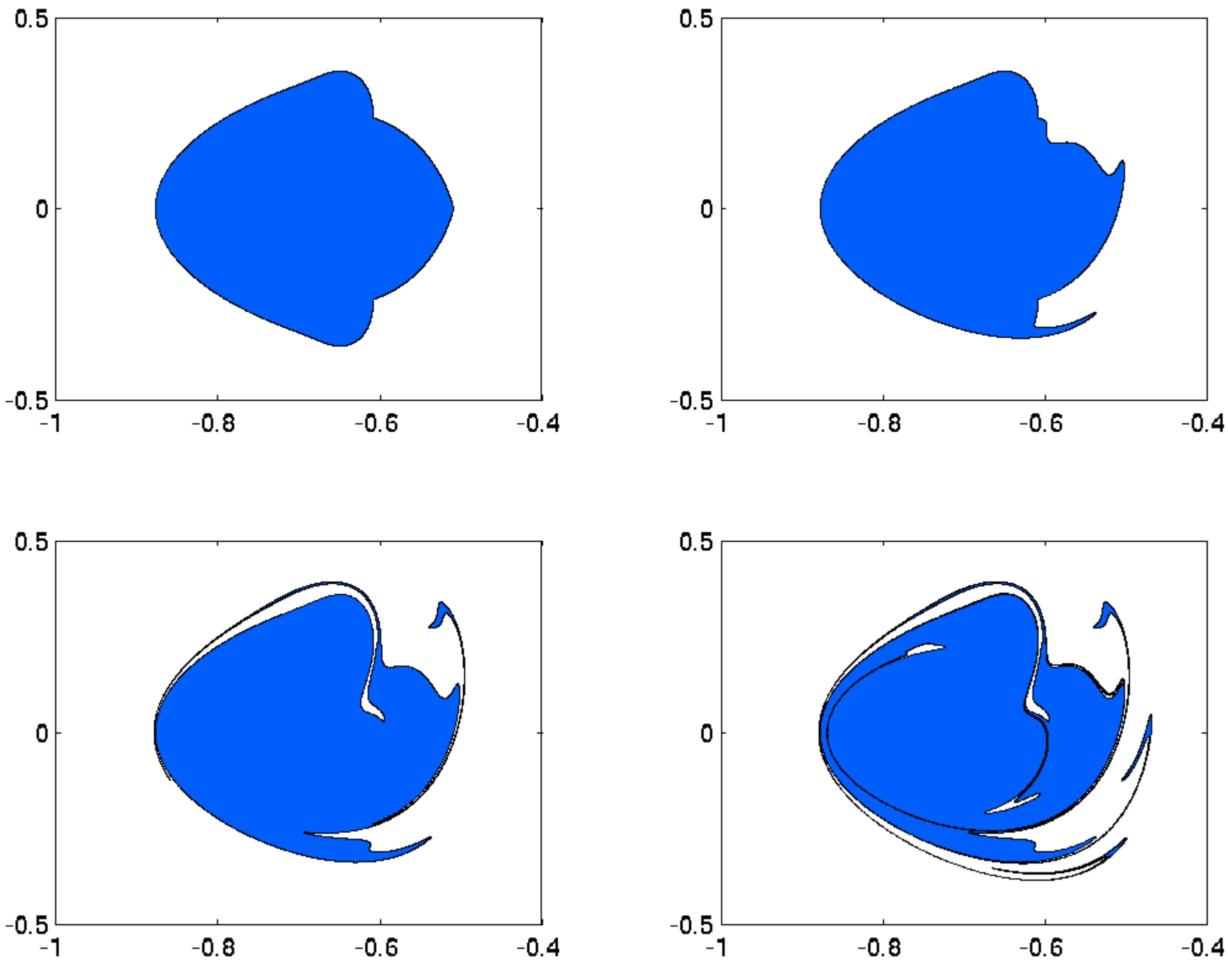

Transport Quantities

□ Quantities of interest:

$$
\begin{aligned}
T_{i,j}(n) &\equiv \text{the total amount of species } S_i \text{ contained in} \\
&\quad\ \text{region } R_j \text{ immediately after the } n\text{-th iterate} \\
&= \mu(f^{-n}(R_j) \cap R_i)
\end{aligned}
$$

Transport Quantities

- **Quantities of interest:**

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$$= \mu(f^{-n}(R_j) \cap R_i)$$

- **Our goal:**

Compute the $T_{i,j}(n)$ up to some $n_{\max}$

Computational Approaches

- Compare & combine two computational approaches

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- 1) **Invariant manifolds** of fixed points, lobe dynamics; co-dimension one objects which bound regions, etc. MANGEN: Manifold Generation, Lekien etc

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- Compare & combine two computational approaches

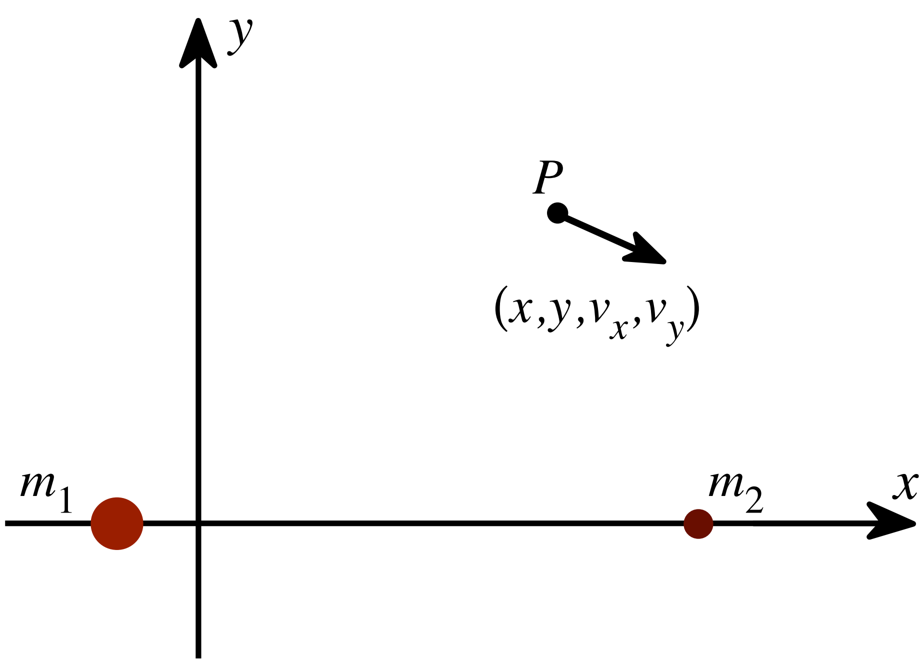
- 1) **Invariant manifolds** of fixed points, lobe dynamics; co-dimension one objects which bound regions, etc. MANGEN: Manifold Generation, Lekien etc

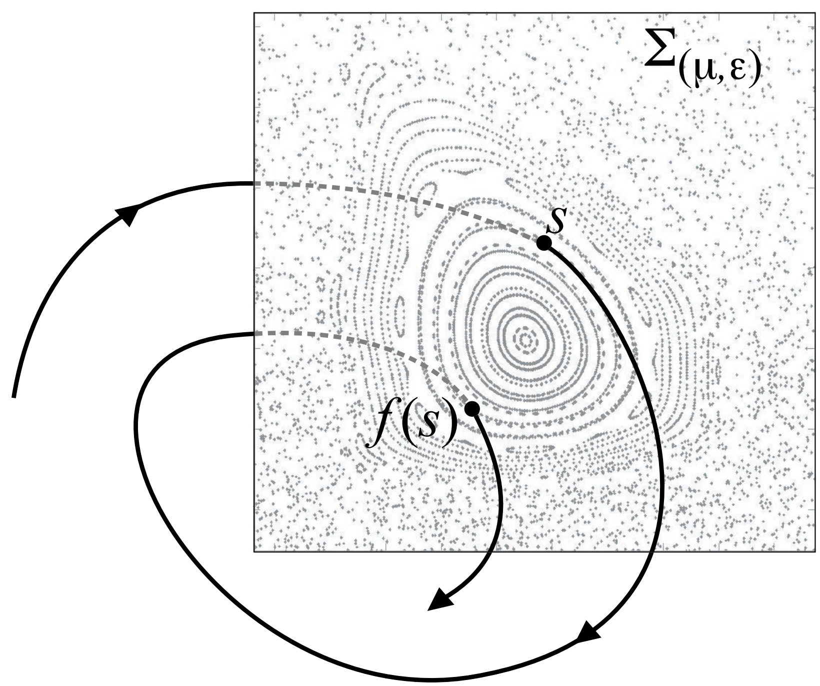
- 2) **Set oriented methods**, almost-invariant sets; direct computation of regions GAIO: Global Analysis of Invariant Objects, Dellnitz etc

Particle in 2-Body Field

- Example problem: test particles in gravity field of two masses, m_1 and m_2, in circular orbit, i.e., the planar, circular restricted three-body problem with $\frac{m_2}{m_1} \approx 10^{-3}$.

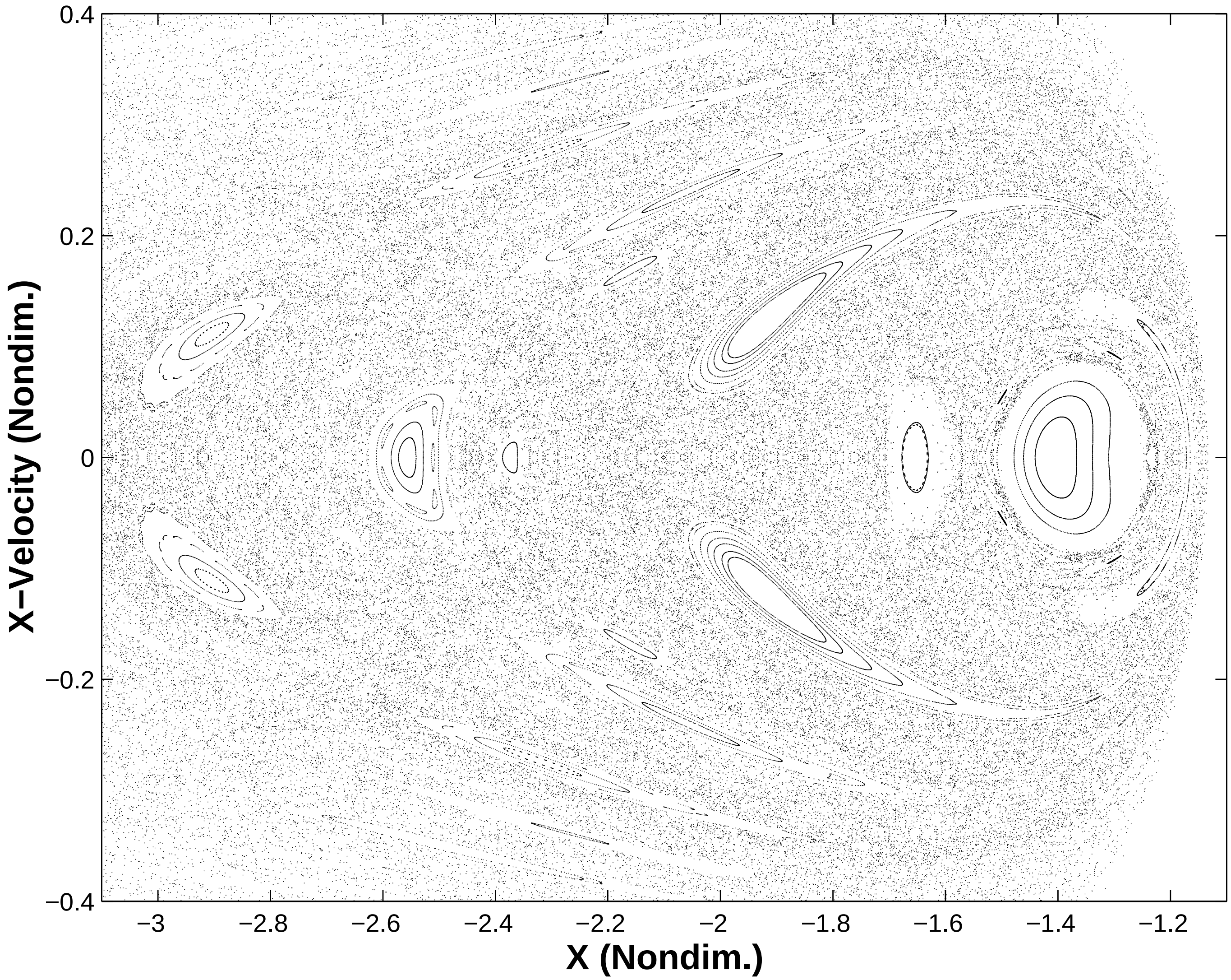
- Reduce to 2D map via Poincaré surface-of-section





Particle in 2-Body Field

- Poincaré map $f : M \to M$ has regular and irregular components. Large connected irregular component, the **"chaotic sea."**



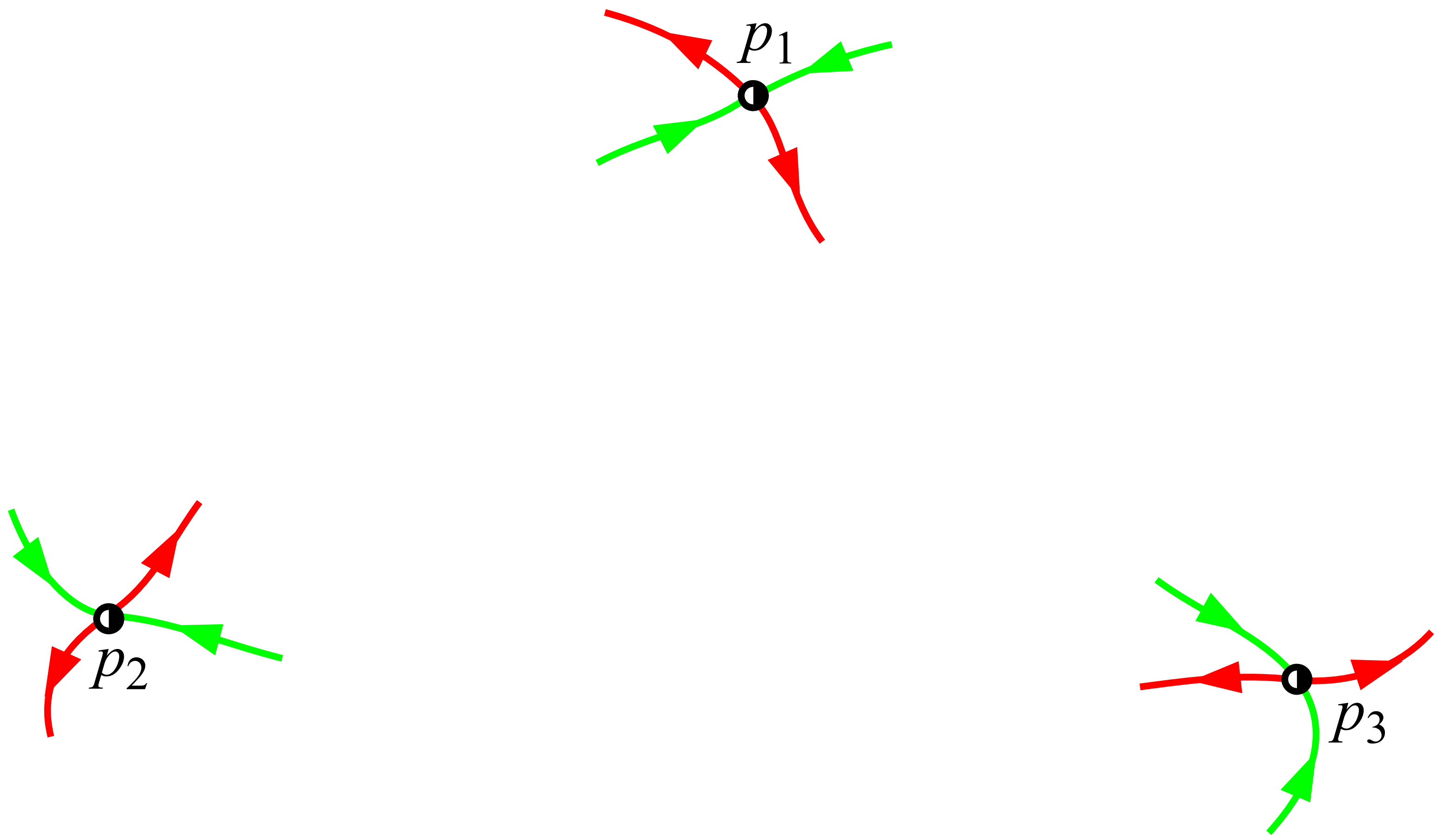
Transport on M

- To understand the transport of points under the Poincaré map f, we consider the **invariant manifolds of unstable fixed points**

- Let $p_i, i = 1, ..., N_p$, denote a collection of saddle-type hyperbolic fixed points for f.

Transport on M

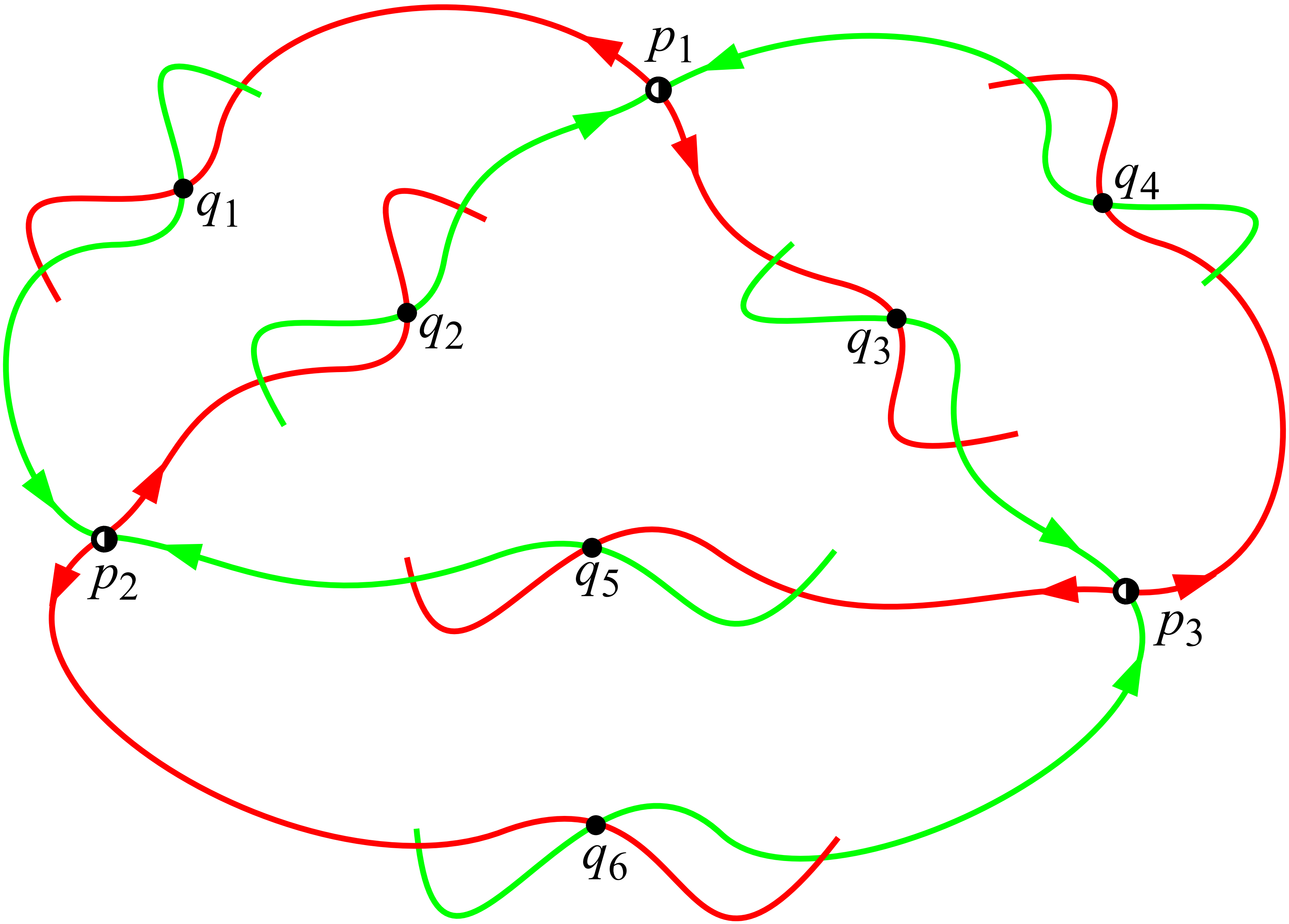
- Local pieces of unstable and stable manifolds



Unstable and stable manifolds in **red** and **green**, resp.

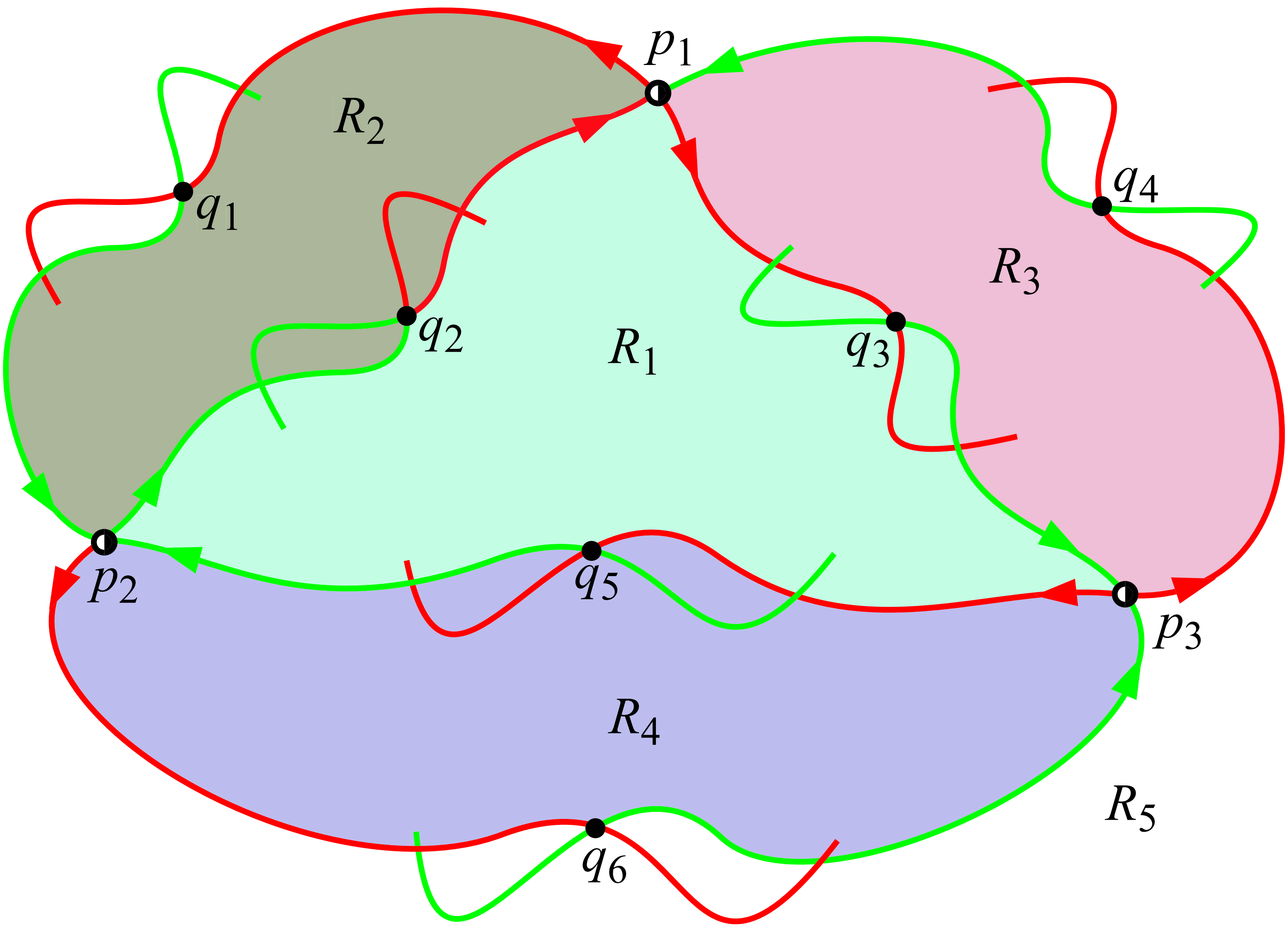
Transport on M

- Intersection of unstable and stable manifolds define **boundaries**.



Transport on M

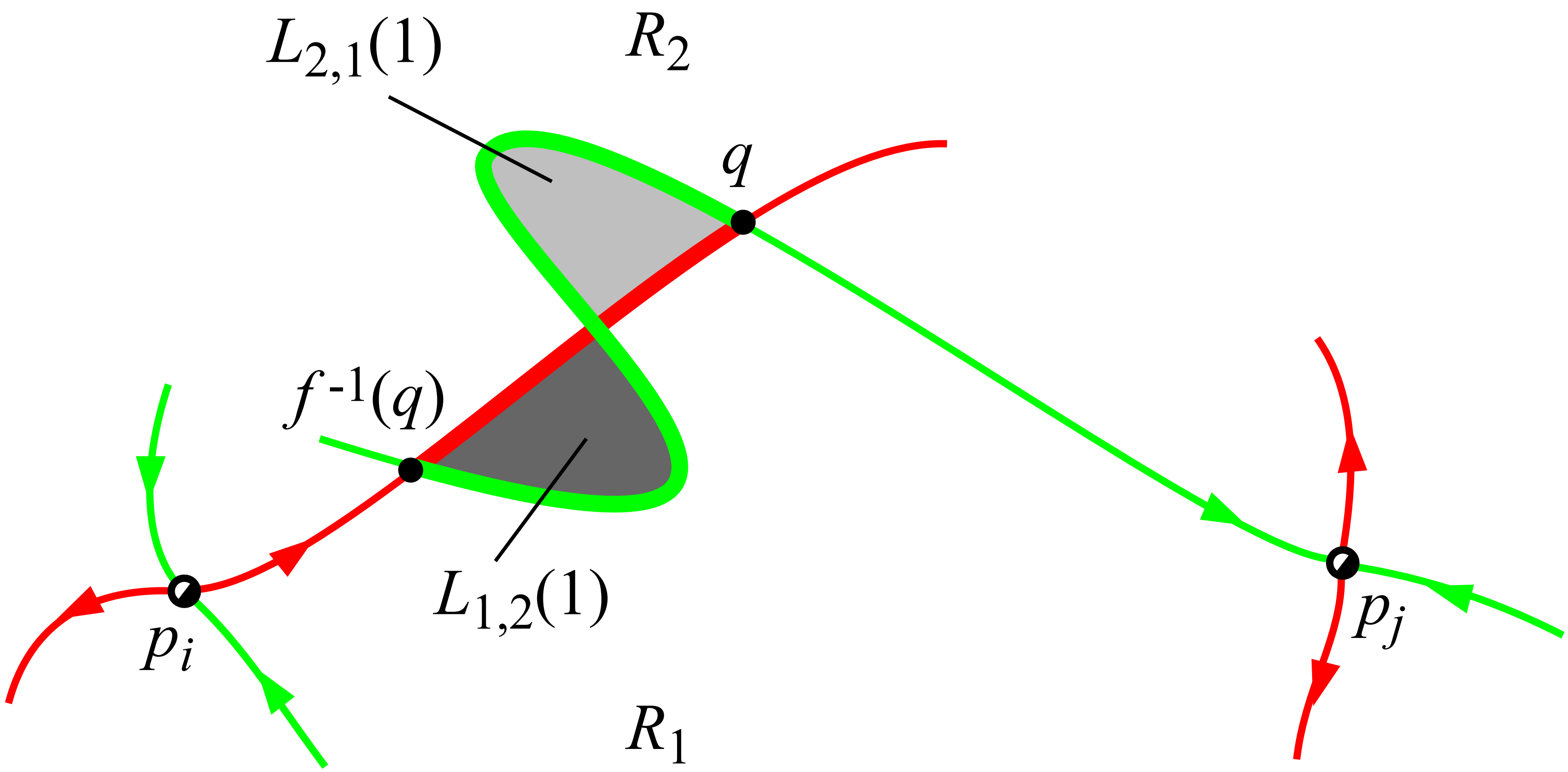
- These boundaries divide phase space into **regions**, $R_i, i = 1, \ldots, N_R$



Lobe Dynamics

□ Local transport: across a boundary

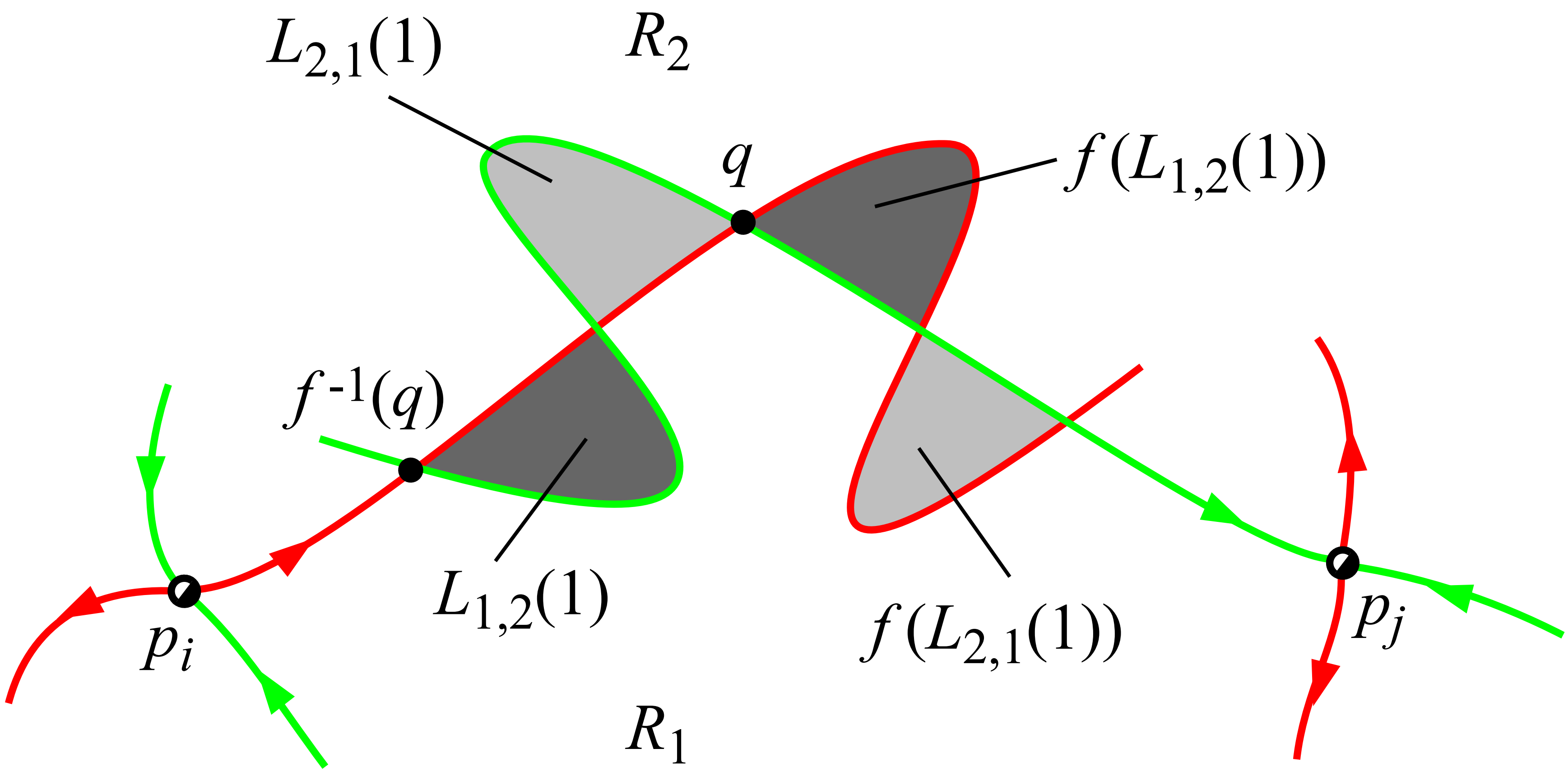
consider small sets bounded by stable & unstable mfds



Lobe Dynamics

- They map from entirely in one region to another under one iteration of f

  $L_{1,2}(1)$ and $L_{2,1}(1)$ are called turnstile **lobes**



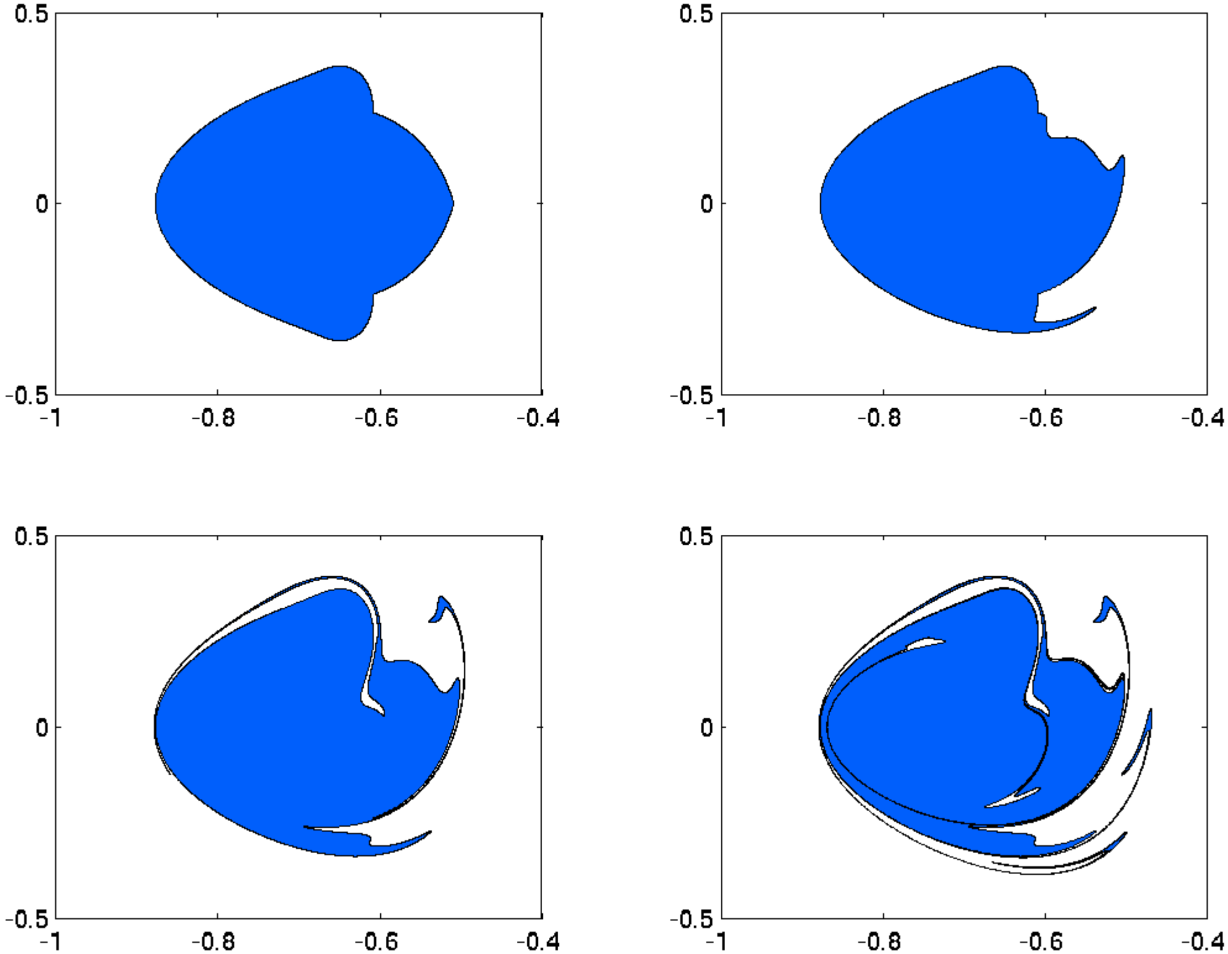
Lobe Dynamics

- **MANGEN:** evolution of a lobe of species S_1 into R_2

insert S1 into R2 movie

Lobe Dynamics

- **Global transport** between regions $(T_{i,j}(n))$ is completely described by the dynamical evolution of lobes.



Set Oriented Methods

Overview

- Partition phase space into **loosely coupled regions**

$$R_i, i = 1, \ldots, N_R,$$

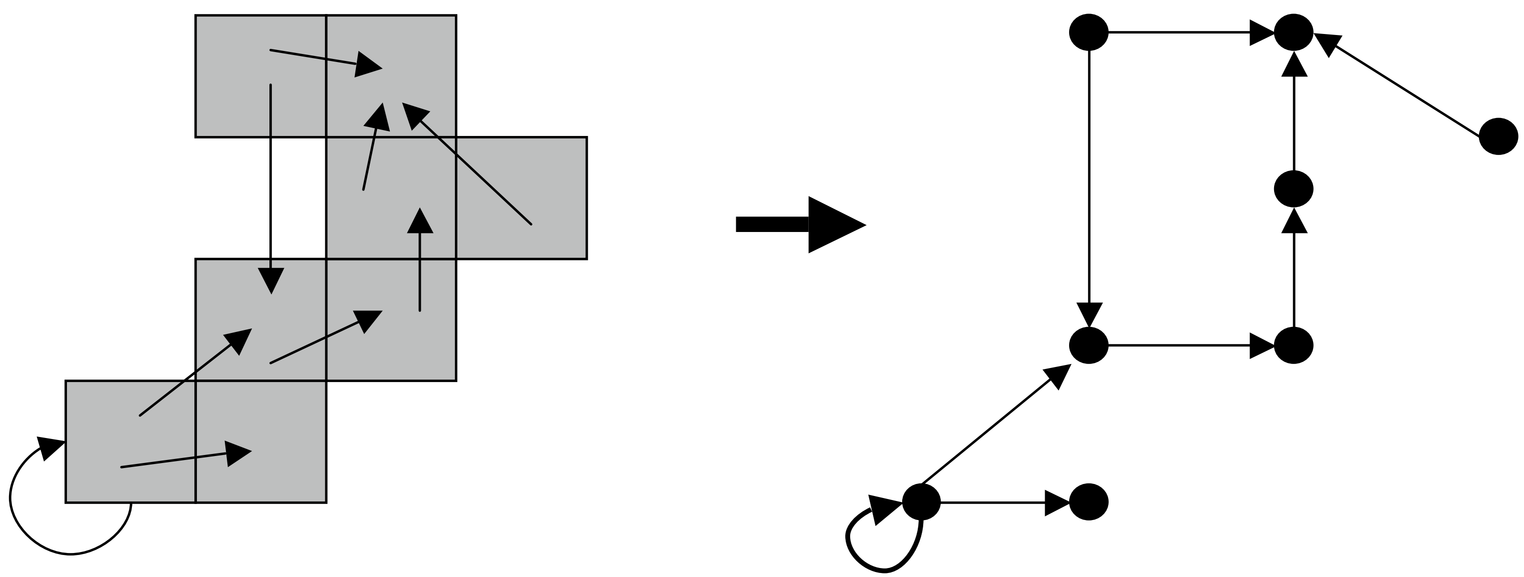
- Probability is small for a point in a region to leave in a short time under f.

- These **almost-invariant sets** (AIS's) define macroscopic structures preserved by the dynamics.

- The transport, $T_{i,j}(n)$, between almost-invariant sets can then be determined.

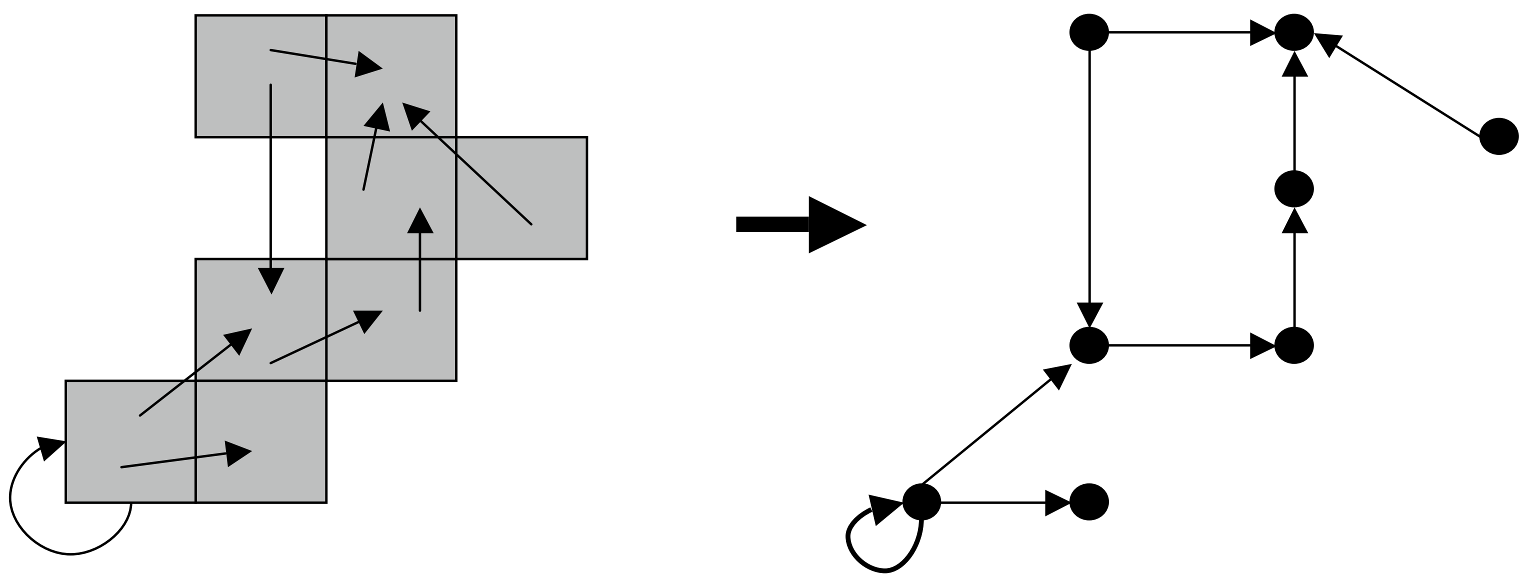
Almost-Invariant Sets

- 1) discretize the phase space into boxes; model boxes as the vertices and transitions between boxes as edges of a directed graph



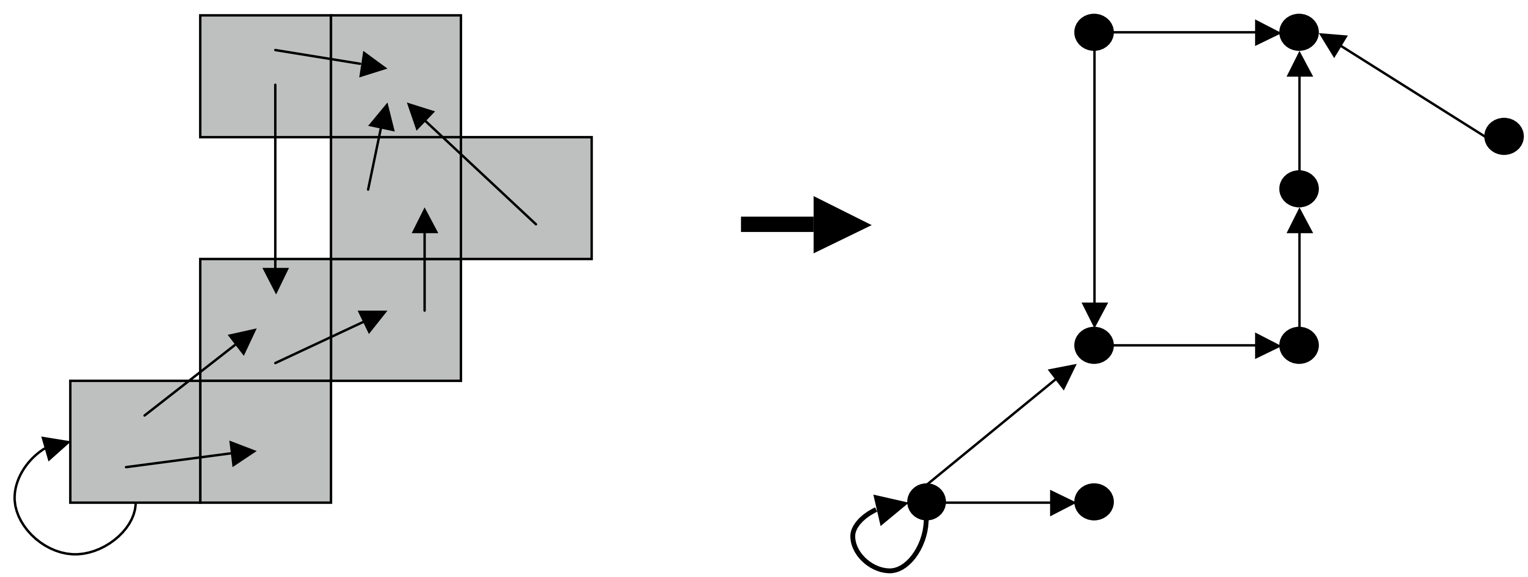
Almost-Invariant Sets

- 2) use graph partitioning methods to divide the vertices of the graph into an optimal number of parts such that each part is highly coupled within itself and only loosely coupled with other parts



Almost-Invariant Sets

- 3) by doing so, we can obtain AIS's and analyze transport between them



Almost-Invariant Sets

Box Formulation

- Create a fine box partition of the phase space $\mathcal{B} = \{B_1, \ldots B_q\}$, where q could be 10^7+
- Consider a (weighted) q-by-q **transition matrix**, P, for our dynamical system, where

$$P_{ij} = \frac{\mu(B_i \cap f^{-1}(B_j))}{\mu(B_i)},$$

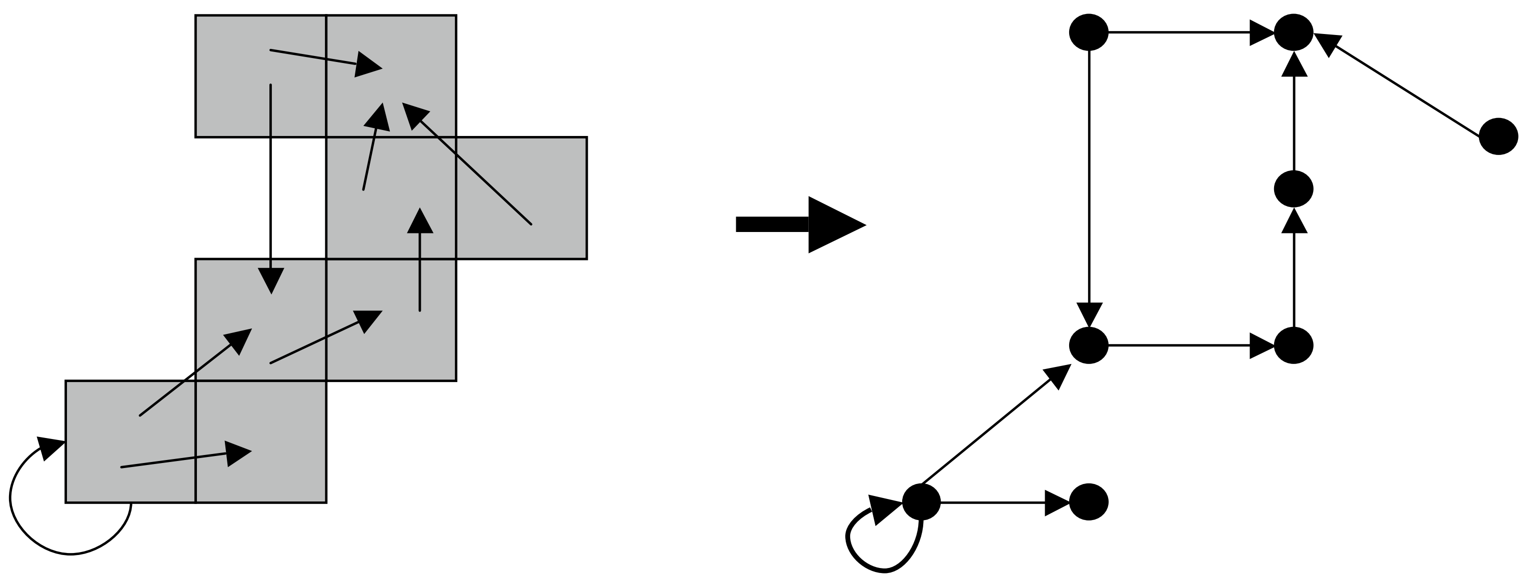
the *transition probability* from B_i to B_j

- P is an approximation of our dynamical system via a finite state Markov chain.

Almost-Invariant Sets

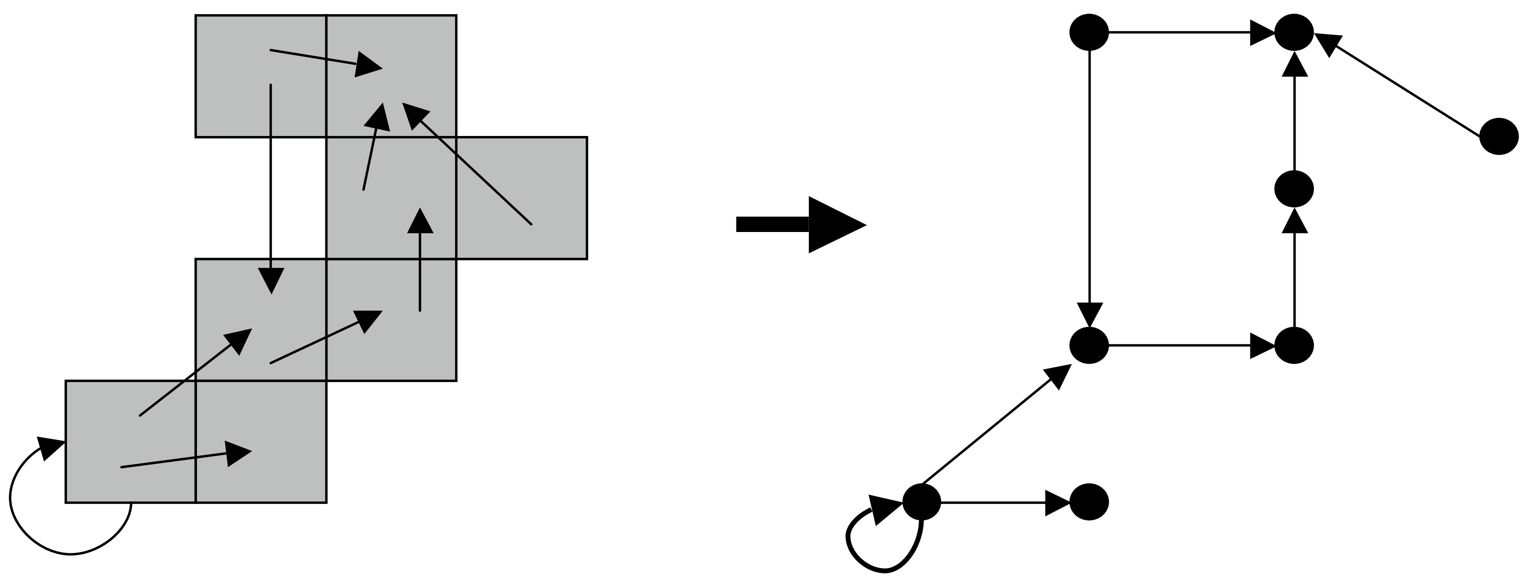
Graph Formulation and Partitioning

- P has a corresponding graph representation where nodes of the graph correspond to boxes B_i.



Almost-Invariant Sets

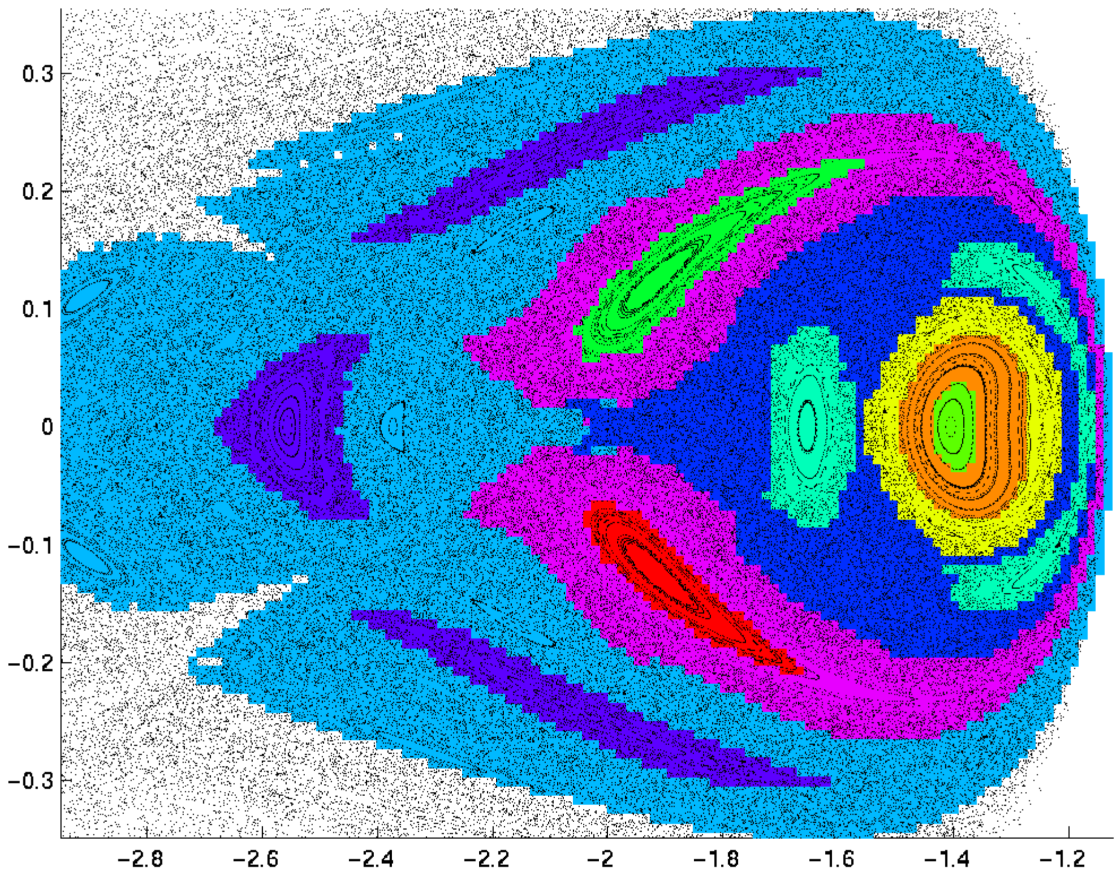
- If $P_{ij} > 0$, then there is an edge between nodes i and j in the graph with weight P_{ij}.
- Partitioning into AIS's becomes a problem of finding a minimal cut of this graph.



Almost-Invariant Sets

□ **AIS's correspond with key dynamical features**

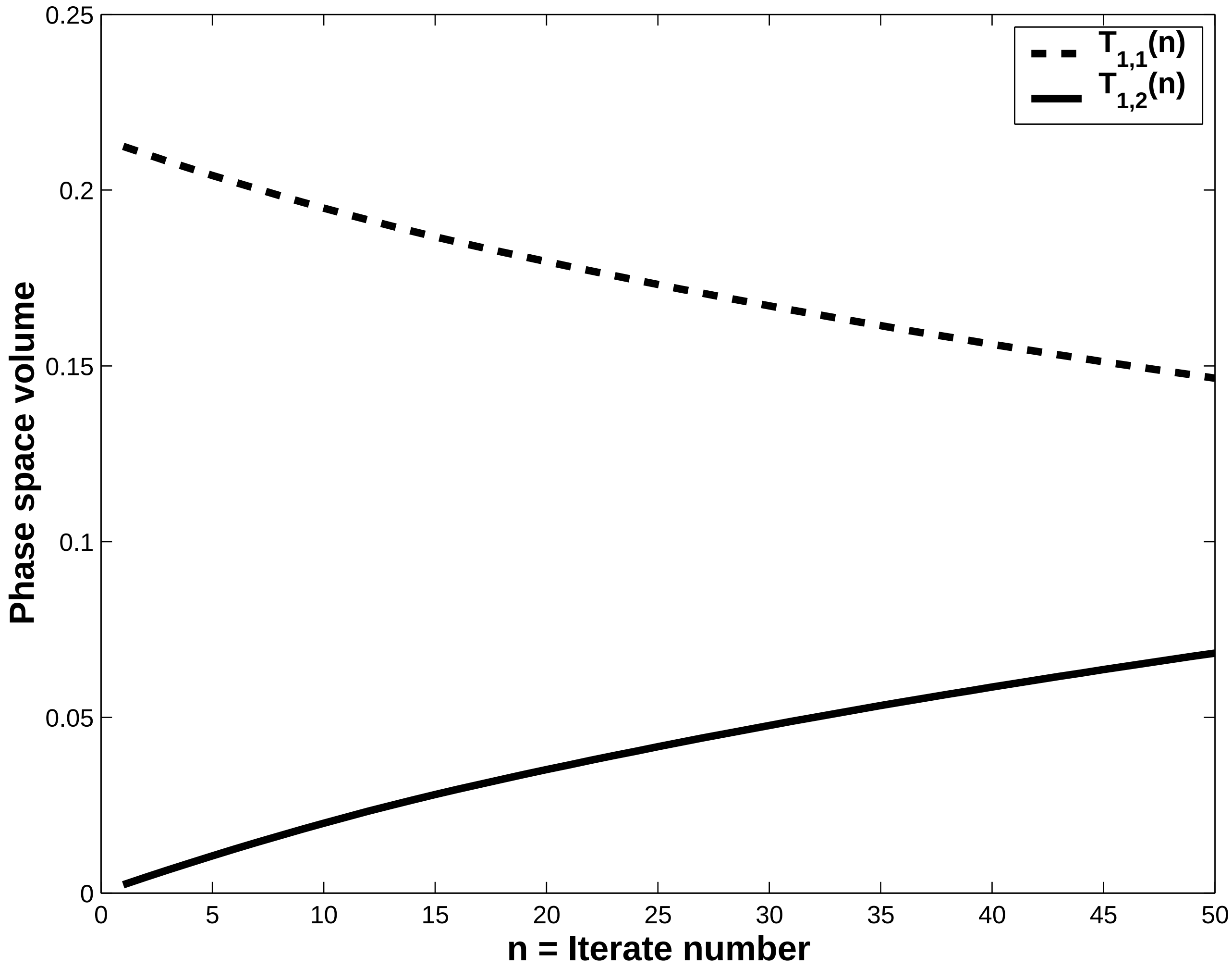
More refined methods like MANGEN can pick up details



The phase space is divided into several invariant and almost-invariant sets.

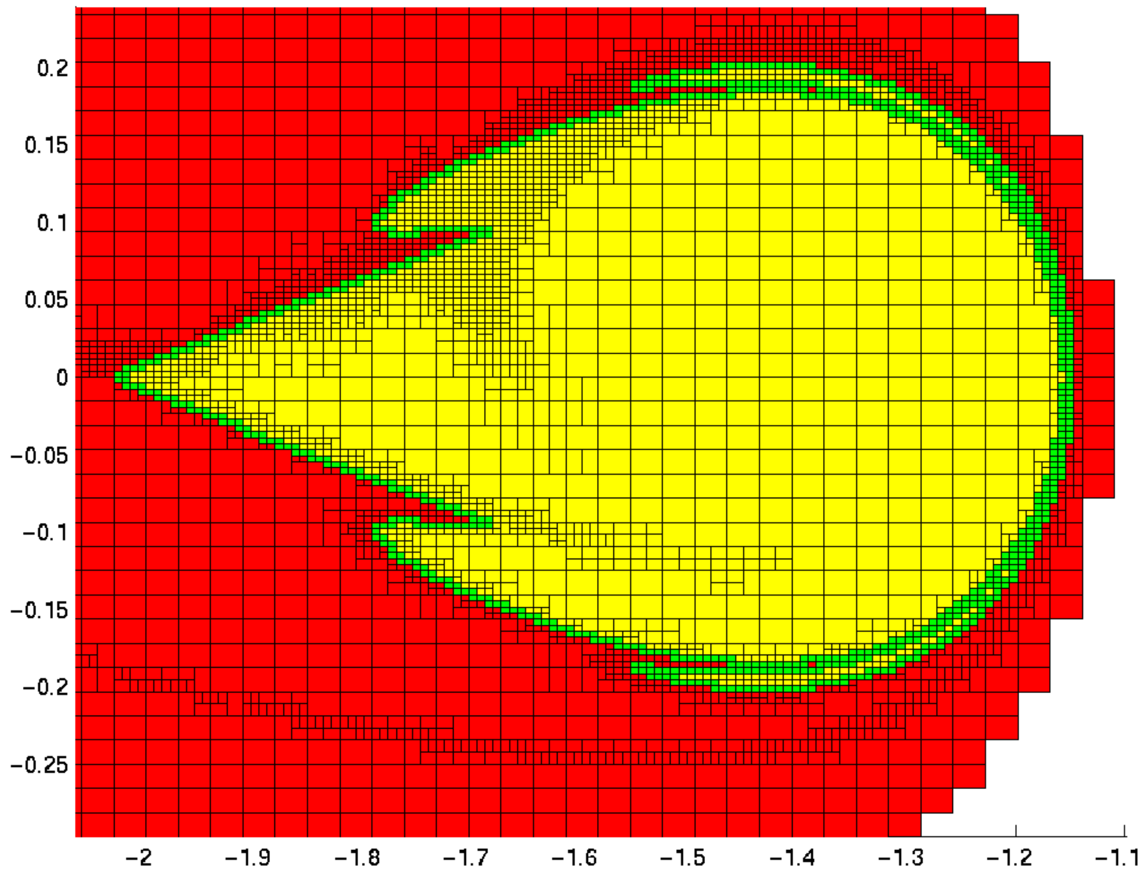
Almost-Invariant Sets

- Using the box formulation and GAIO, the $T_{i,j}(n)$ can be computed for large n. Agrees with MANGEN result.



Almost-Invariant Sets

- To speed the computation, box refinements are performed where transport related structures, e.g., lobes, are located.



Summary & Conclusion

- The merging of **statistical** and **geometric** approaches yields a very powerful tool.

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- Example problem: restricted 3-body problem.
- Both find the same regions

  AIS's, statistical features, are identified with regions, geometric features

Summary & Conclusion

- Theoretically, transport between regions determined by images and pre-images of lobes.

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- GAIO uses transition matrix between many boxes; Resolution limited by max box number; aided by box refinement along lobes $n_{\max} \approx 50$

Summary & Conclusion

- Theoretically, transport between regions determined by images and pre-images of lobes.
- MANGEN computes only region & lobe boundaries; Stretching of boundaries, memory limits;

  $n_{\max} \approx 20$
- GAIO uses transition matrix between many boxes; Resolution limited by max box number; aided by box refinement along lobes

  $n_{\max} \approx 50$
- Same values for $T_{i,j}(n)$ over common time window.

Future Directions

- AIS & lobe dynamics in 3D+, e.g., astronomy, chemistry

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  GAIO for coarse picture, transport calculations

  MANGEN to refine on regions of interest

  $\Rightarrow$ important for precision navigation

Future Directions

- AIS & lobe dynamics in 3D+, e.g., astronomy, chemistry
- AIS for time dependent systems? e.g., ocean dynamics
- **Numerical algorithms are crucial**:
  GAIO for coarse picture, transport calculations
  MANGEN to refine on regions of interest
  $\Rightarrow$ important for precision navigation
- **Merge techniques into single package**:
  Box formulation, graph algorithms
  Co-dimension one objects
  Adaptive conditioning based on curvature

Selected References

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http://www.cds.caltech.edu/∼shane
http://www.nast-group.caltech.edu/

The End

Typesetting Software: TeX, ***Textures***, LaTeX, `hyperref`, `texpower`, Adobe Acrobat 4.05
Graphics Software: Adobe Illustrator 10.0.1
LaTeX Slide Macro Packages: Wendy McKay, Ross Moore