Dynamical Systems and Space Mission Design

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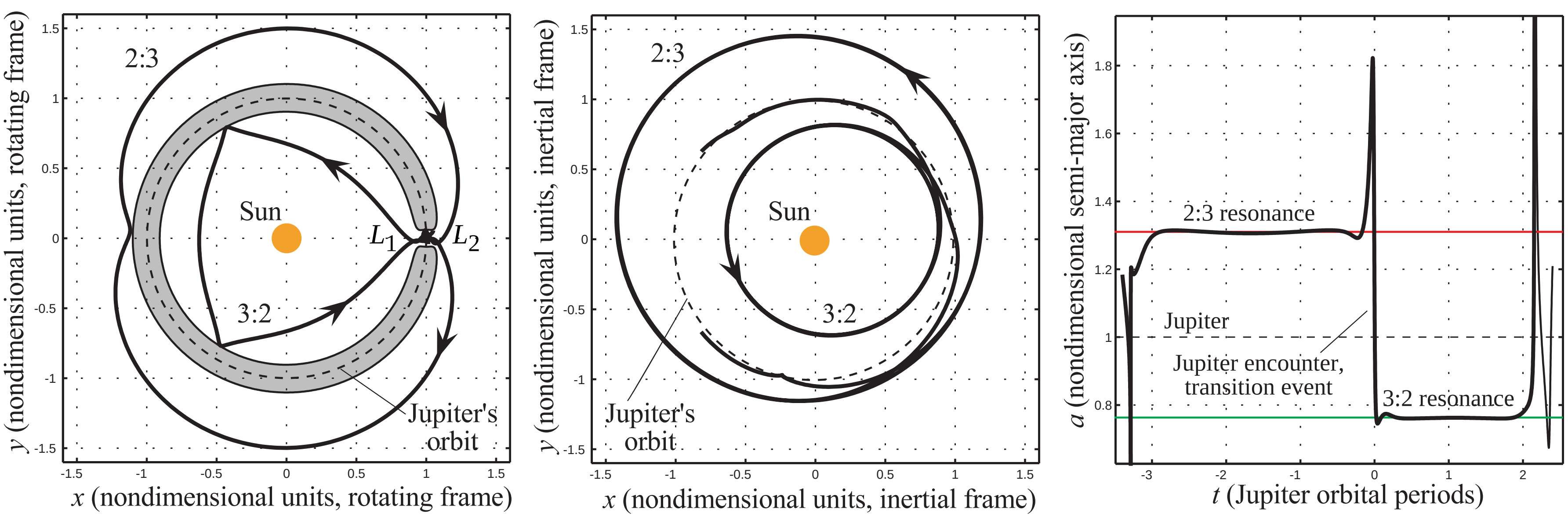
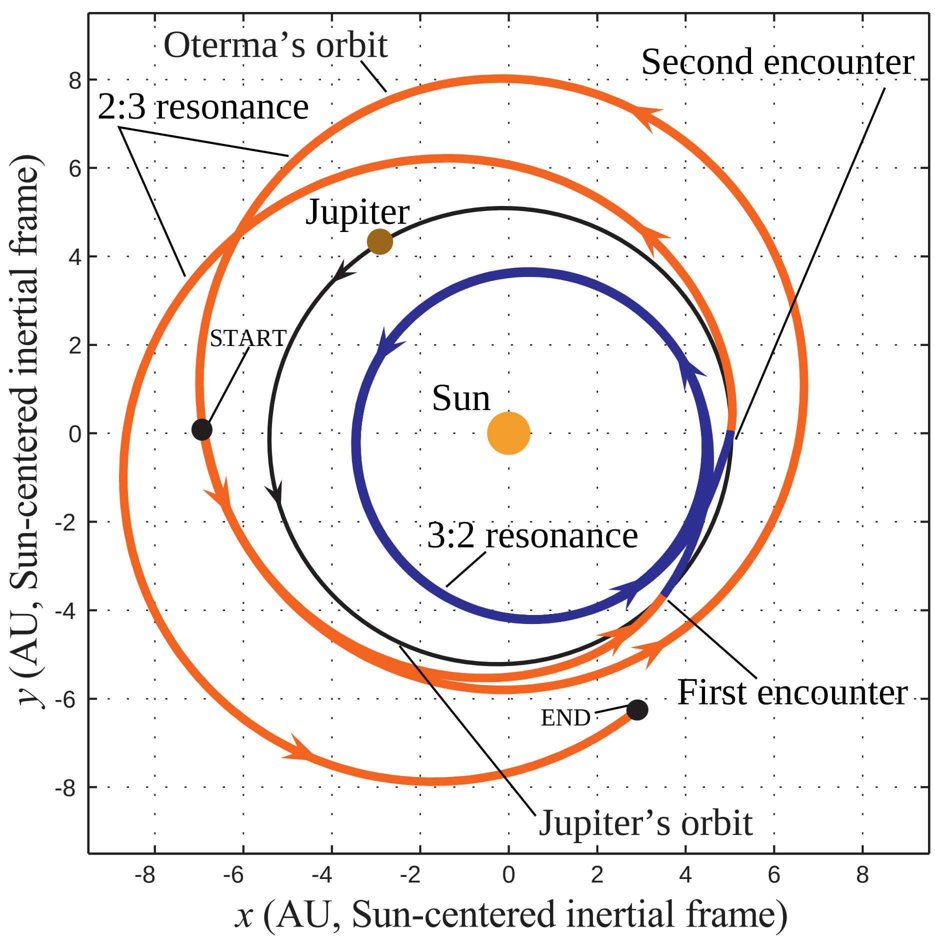
Resonance Transition of Comets: Outline

- Outline of Lecture 4B:
  - **Resonance transition** seen in comets such as ***Oterma***.
  - Mixed phase space of 3-body problem:
  - Mean motion resonance **"islands"** imbedded in chaotic **"sea."**
  - Exterior and interior resonances connected by Lyapunov orbit stable & unstable manifold tubes, the ***dynamical channels***.
  - **Future work**: transition between planets, belts, etc.



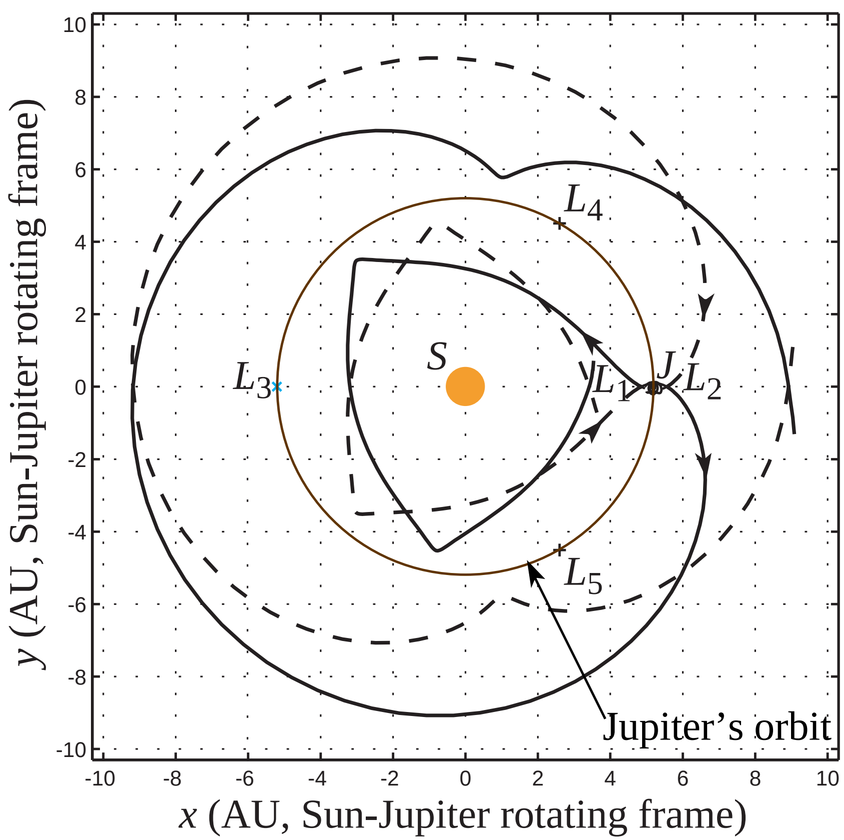
Jupiter Comets: *Oterma*

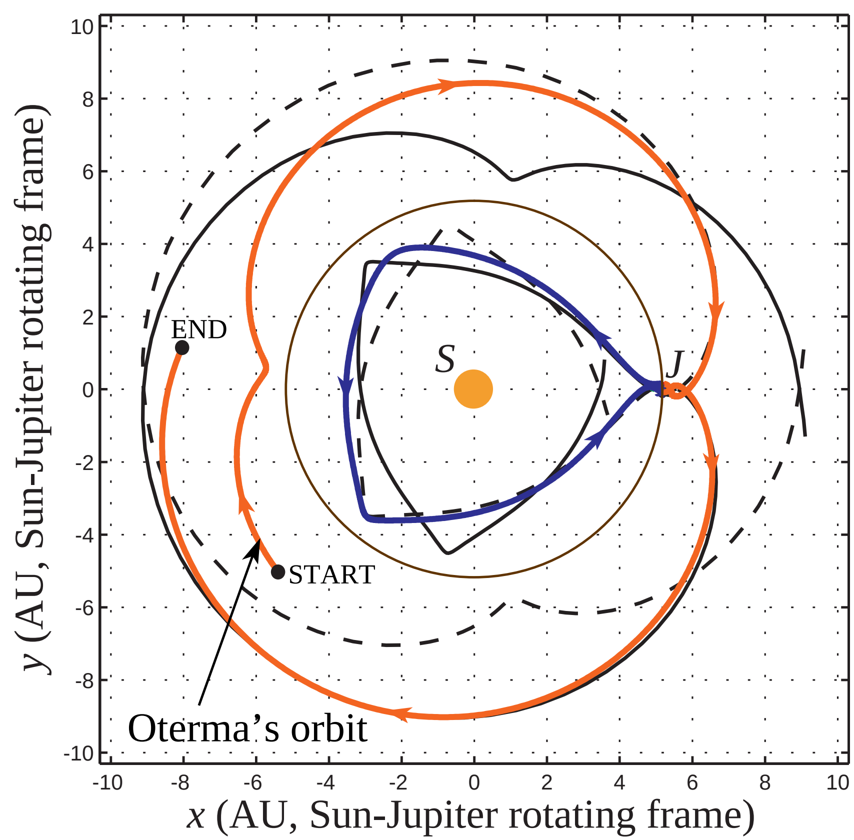
- Some Jupiter comets perform a **rapid transition** from the **outside** to the **inside** of Jupiter's orbit.
- **Captured temporarily** by Jupiter during transition.
- **Exterior** (**2:3 resonance**). **Interior** (**3:2 resonance**).



Jupiter Comets: Previous Works

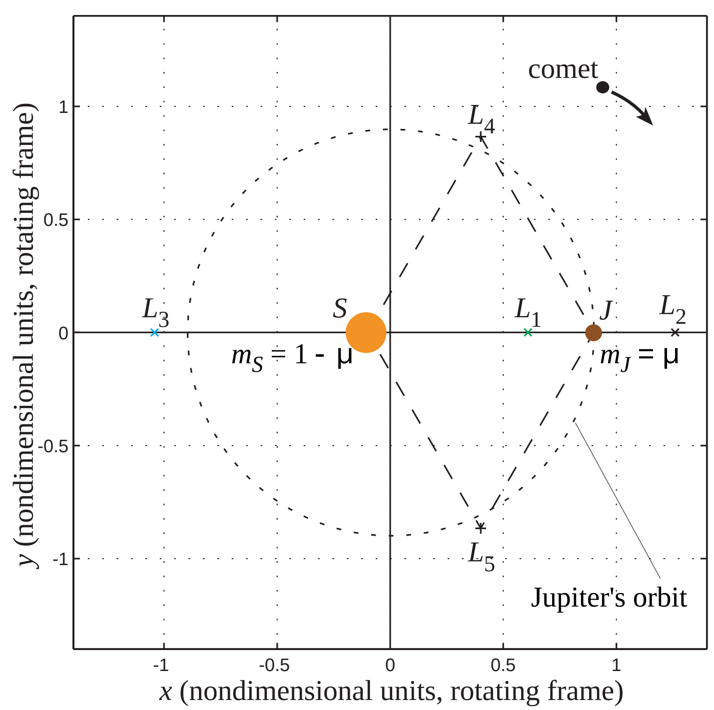
- Belbruno/B. Marsden [1997]
- Lo/Ross [1997] :
  - Jupiter comets *Oterma*, *Gehrels 3*, etc. in Sun-Jupiter **rotating frame** follow **stable and unstable invariant manifolds** of the equilibrium points L_1 and L_2.





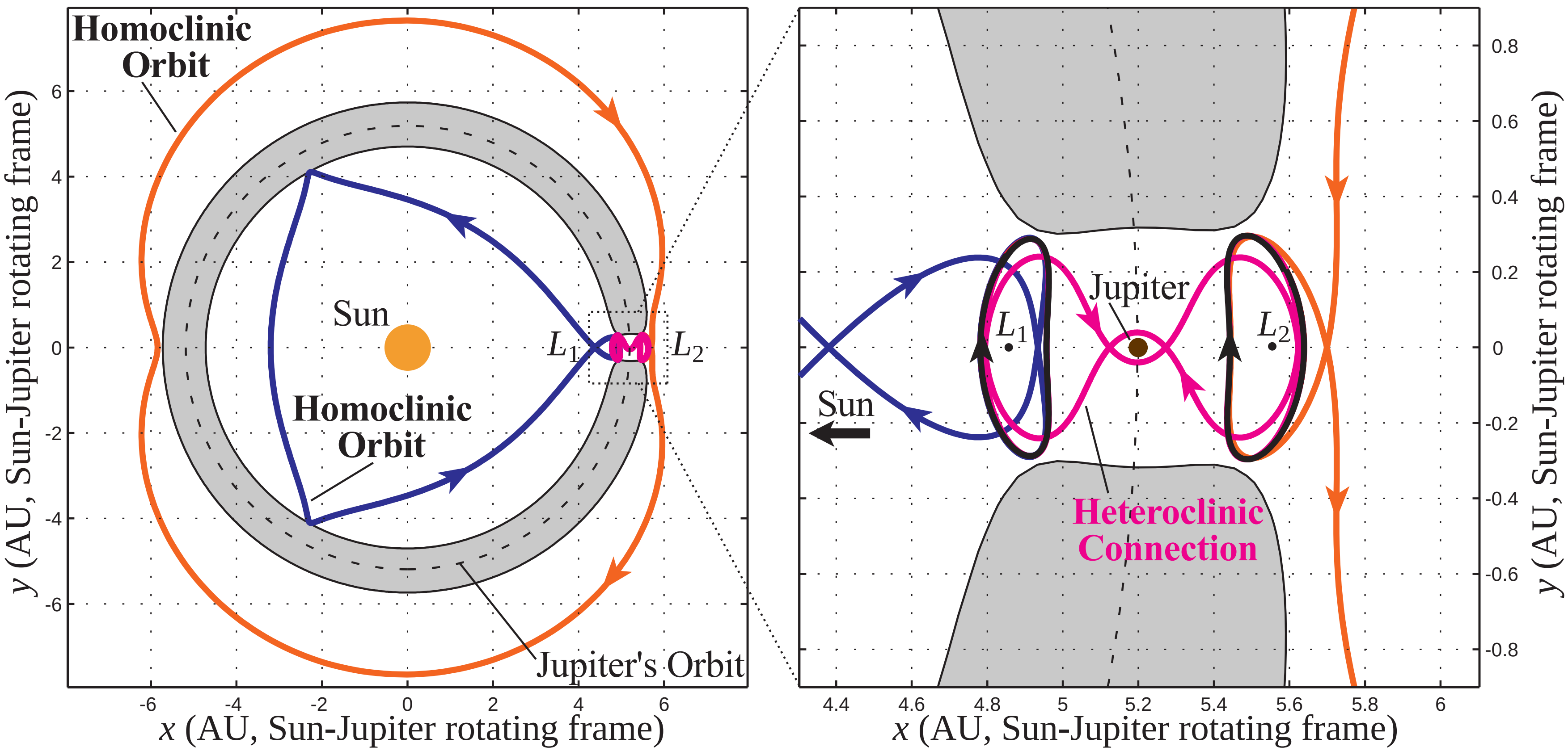
Jupiter Comets: Planar CR3BP Model

- Use planar circular restricted 3-body problem as initial model:
  - **Simplest 3-body model**, easiest to analyze.
  - Comets of interest are mostly **heliocentric**, but their perturbation is dominated by **Jupiter's gravitation**.
  - Their motion is nearly in Jupiter's **orbital plane** ($i < 5°$), and Jupiter's small **eccentricity** (0.0483) plays little role during resonance transition.



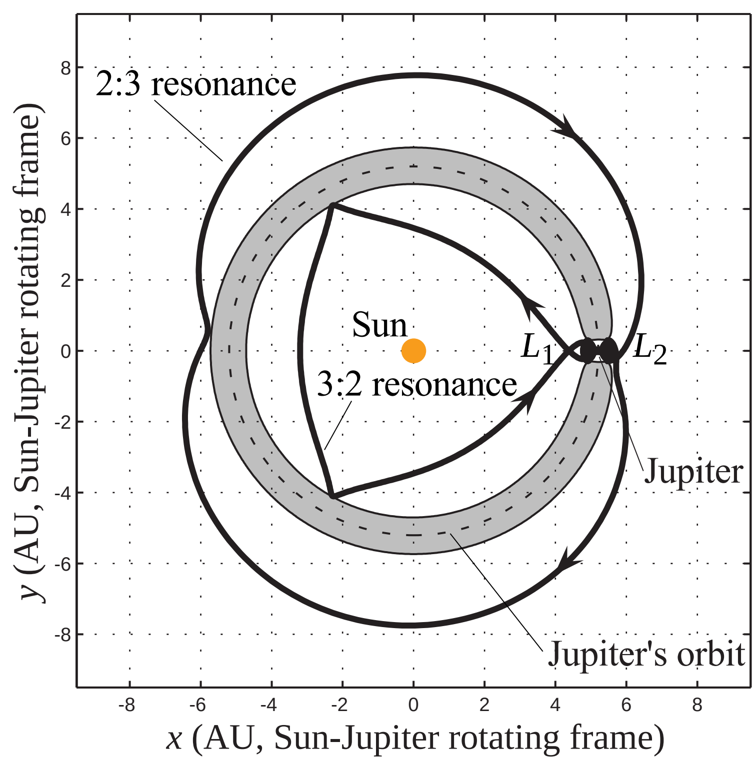
Jupiter Comets: Heteroclinic Connection

- More recently, Koon/Lo/Marsden/Ross [2000]:
  - Found **heteroclinic connection** between pair of periodic orbits.
  - Found a large class of **orbits** near this (homo/heteroclinic) ***chain***.
  - Comets can follow these ***dynamical channels*** in rapid transition between **interior** and **exterior** Hill's regions.

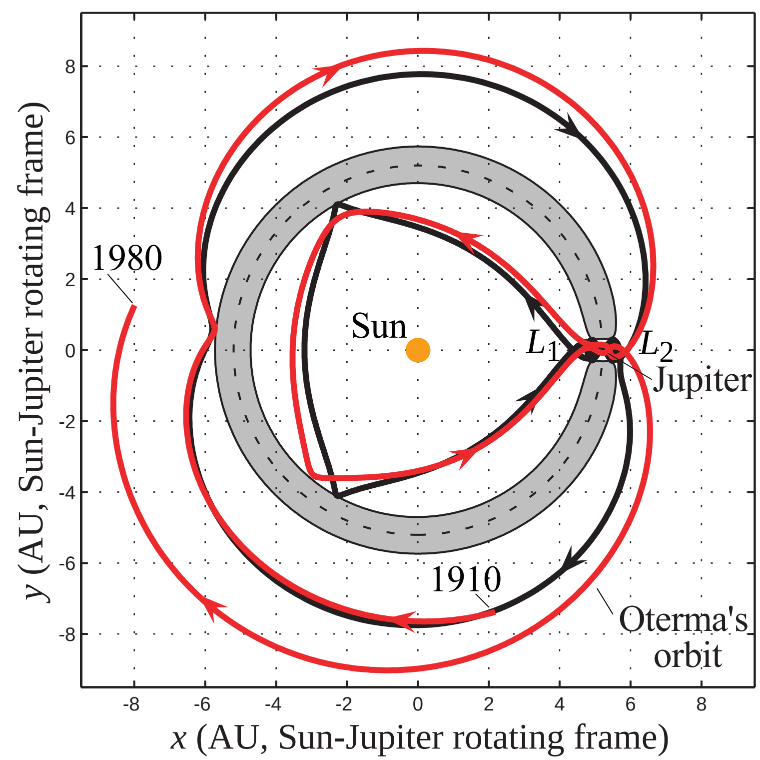


Jupiter Comets: Following Dynamical Channels

- For instance, consider the comet ***Oterma*** from 1910 to 1980.
- The average Jacobi constant for ***Oterma*** during its transition is $C = 3.030 \pm 0.005$ (computed at Jupiter encounter).
- We can compute a **homoclinic-heteroclinic chain** for $C = 3.030$ (shown in **black** on the left).
- Overlaying the chain, we plot ***Oterma's*** orbit in **red** (at right).



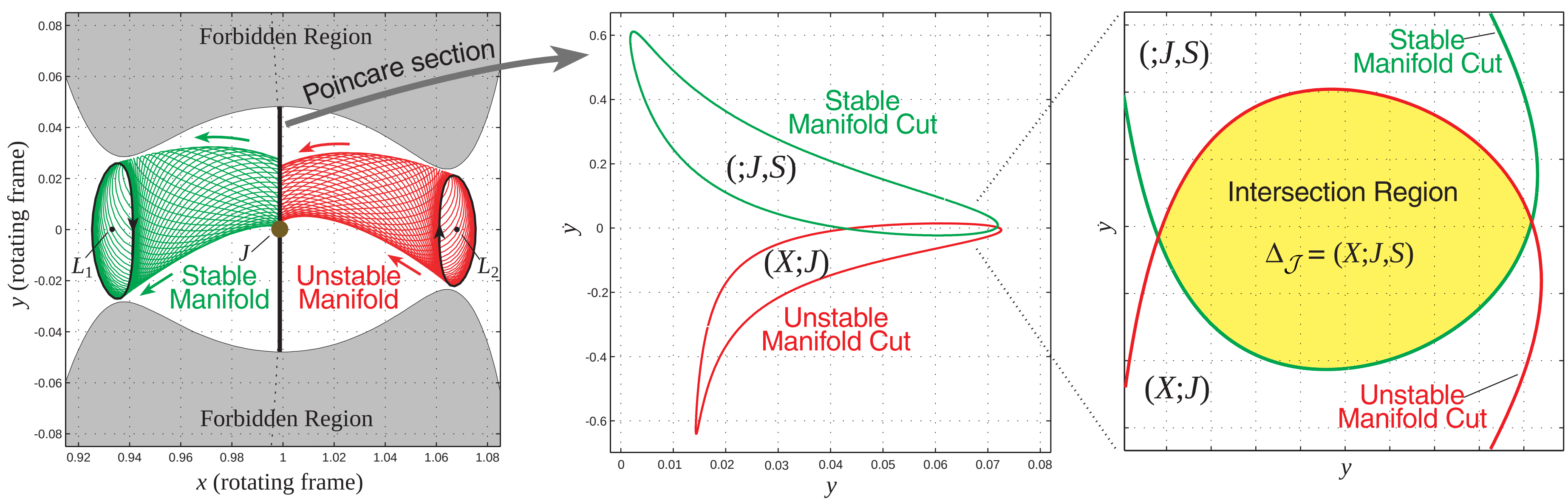
(a)



(b)

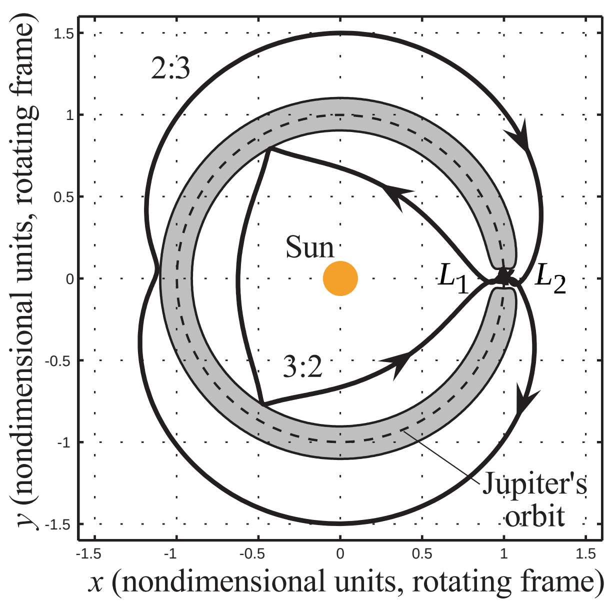
Jupiter Comets: Rapid Transition Mechanism

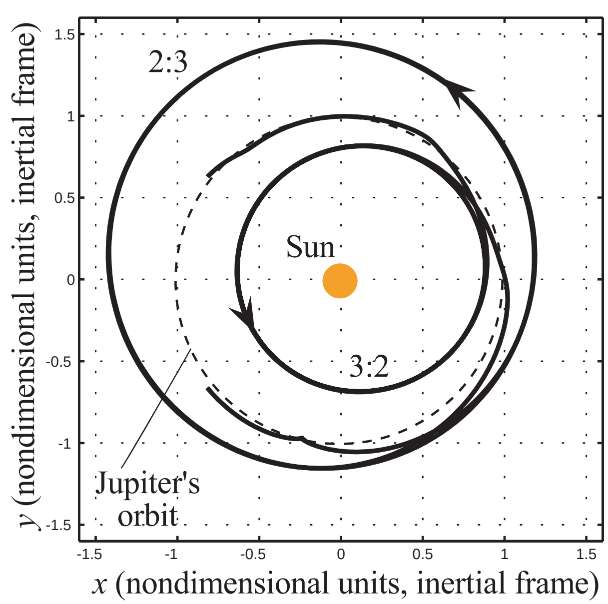
- **Rapid transition** between the interior and exterior regions is possible via the L_1 & L_2 periodic orbit **stable** & **unstable** manifold **tubes** (containing **transit** orbits) and their **intersections**.
- This was a surprising result. Some believed that a third degree of freedom was necessary for such a transition, or that "Arnold diffusion" was invloved.
- But as we have seen, only the **planar CR3BP**, the simplest model of 3-body gravitational dynamics, is necessary.

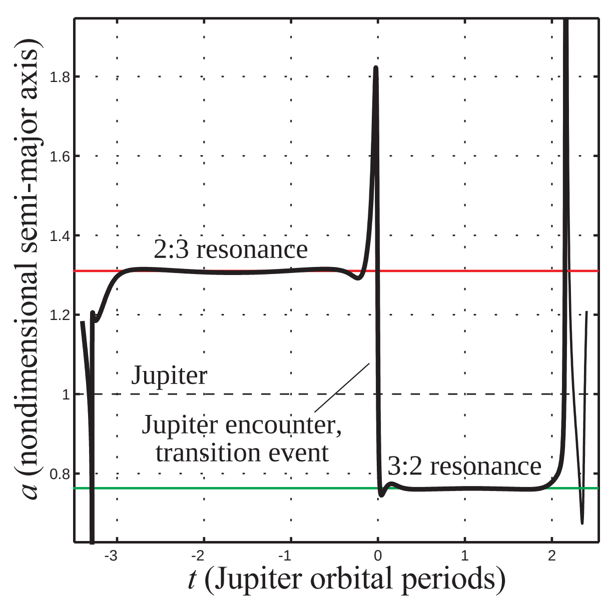


Rapid Transition Mechanism: Resonance Transition

- The tubes are a generic transport mechanism connecting the interior and exterior Hill's regions.
- We wish to understand their role in transport between interior and exterior **mean motion resonances**.
- e.g., we shall try to explain in more precise terms the sense in which ***Oterma*** transitions between the 3:2 and 2:3 mean motion resonances with Jupiter.

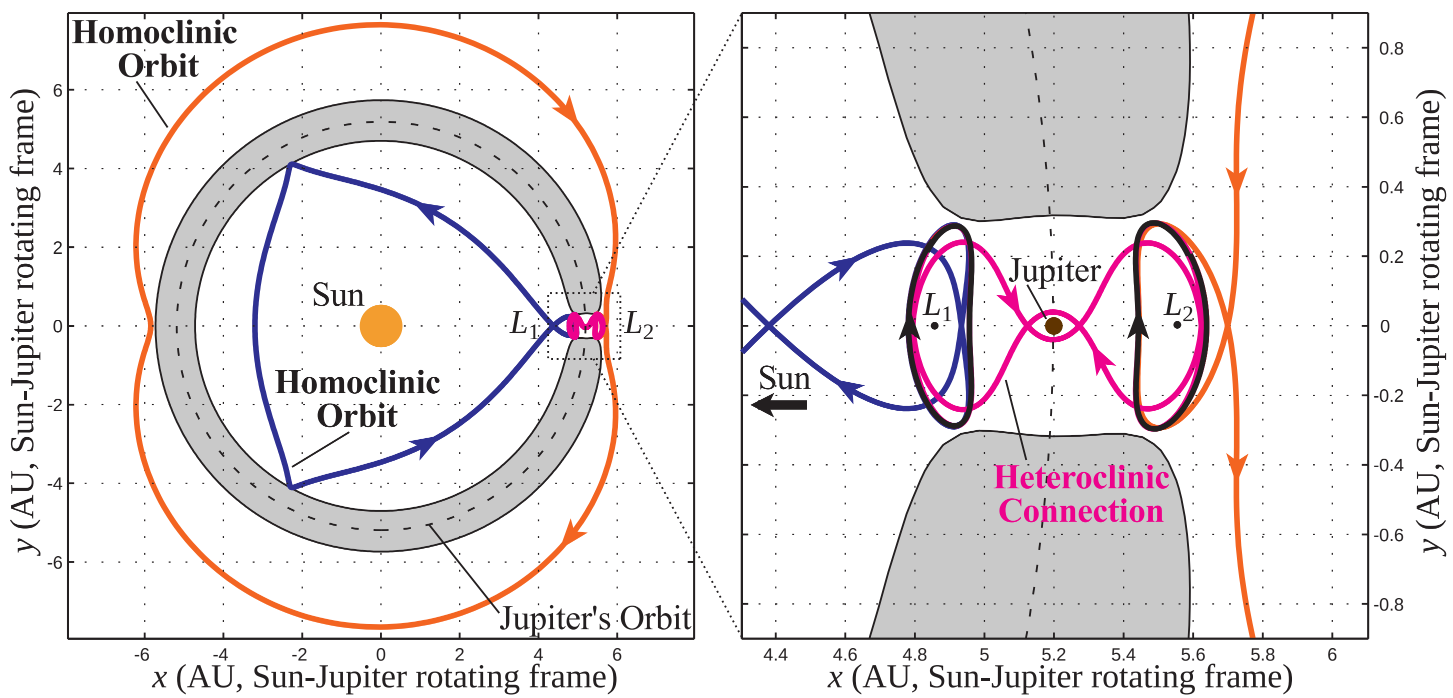






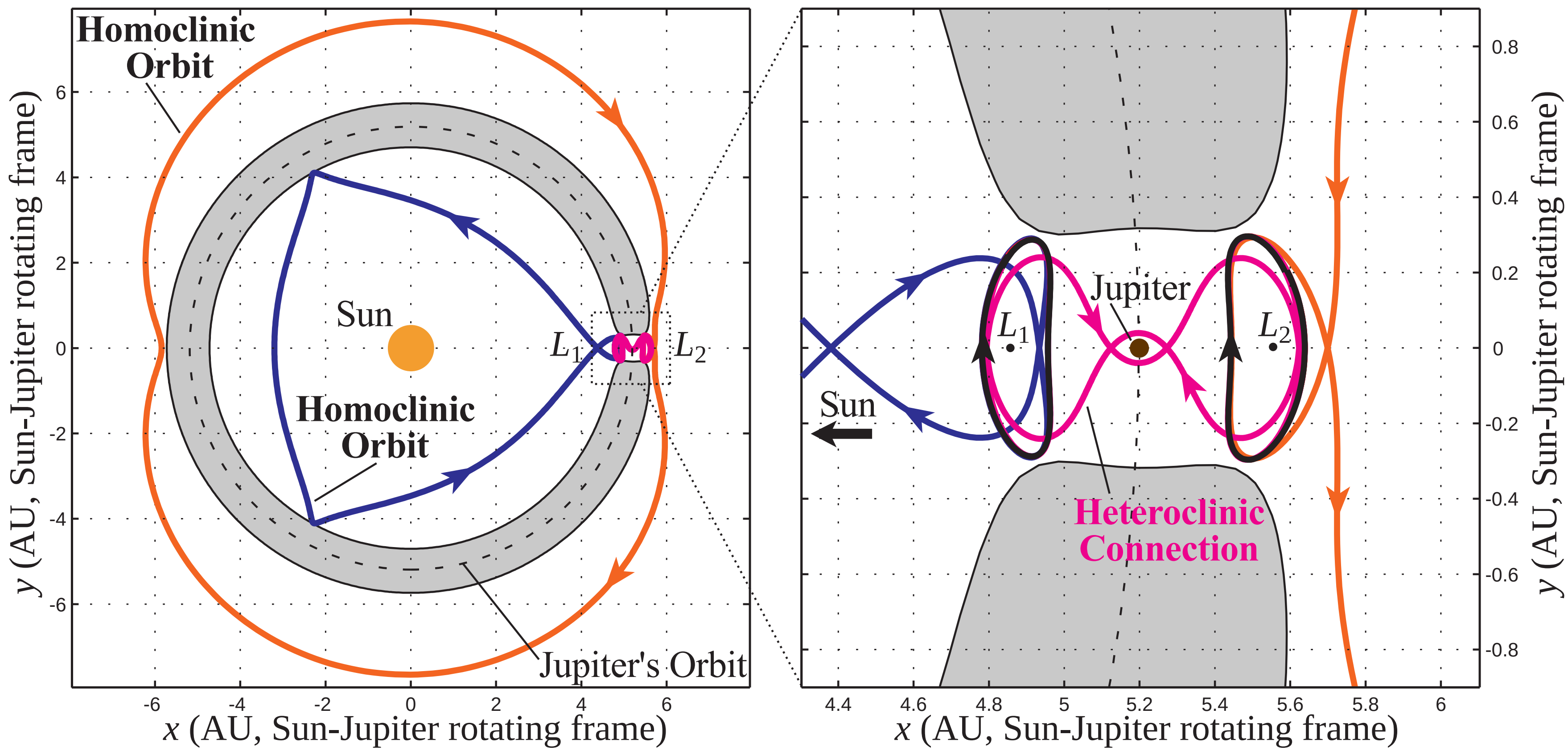
Rapid Transition Mechanism: Resonance Transition

- For the Sun-Jupiter system ($\mu = 0.0009537$), we can construct a homoclinic-heteroclinic chain with Jacobi constant similar to that of ***Oterma*** during its recent Jupiter encounters ($C = 3.030$).
- The chain is a union of orbits: **interior** region orbit homoclinic to L_1 periodic orbit, **exterior** region orbit homoclinic to L_2 periodic orbit, and **heteroclinic connection** between the L_1 & L_2 periodic orbits.



Rapid Transition Mechanism: Resonance Transition

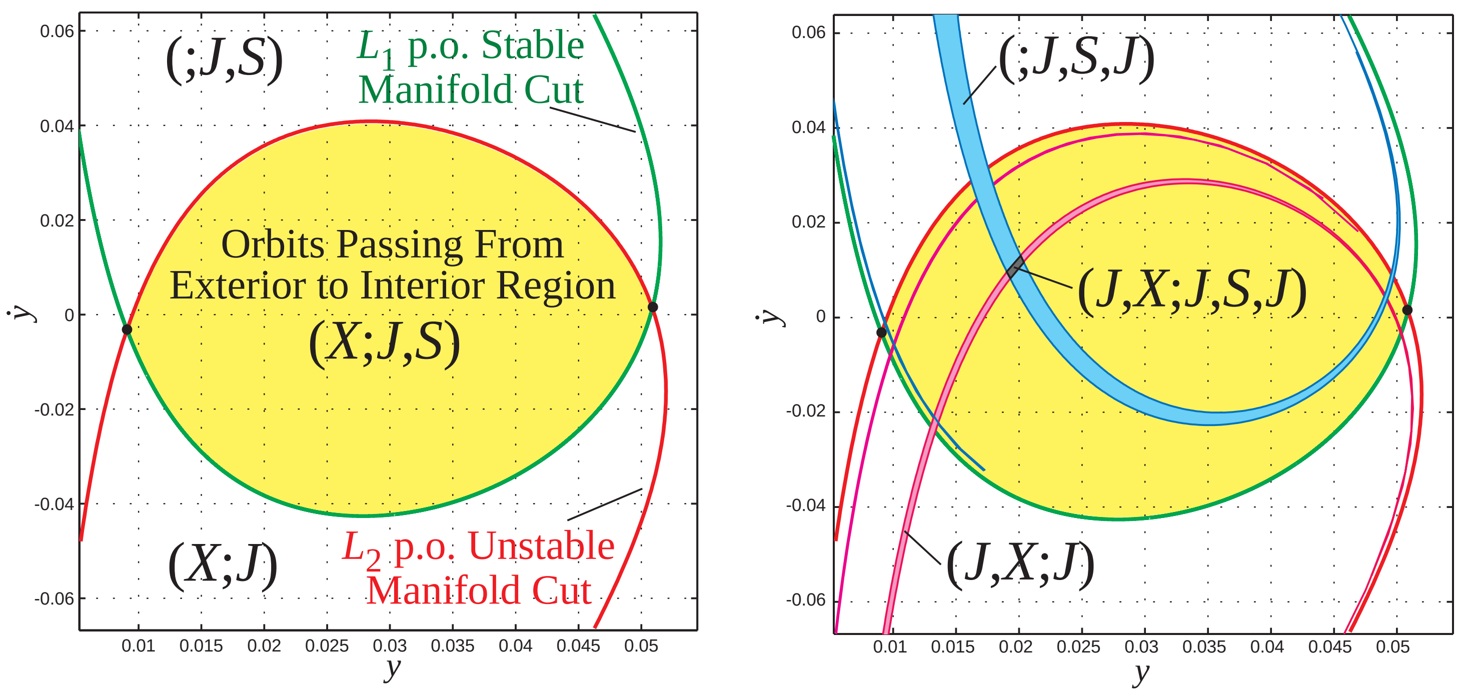
- We choose this chain because its homoclinic orbits are (1,1)-type.
- Limiting to (1,1)-type means, for this particular energy regime, that two different resonance connections are possible; 3:2 to 2:3, and 3:2 to 1:2. This will be explained later. We choose **3:2** to **2:3**, since this matches ***Oterma***'s orbit.



Rapid Transition Mechanism: Resonance Transition

- Our ***main theorem*** tells us that in the vicinity of this chain, there exists an orbit whose symbolic sequence $(\ldots, J, X, J, S, J, \ldots)$ is periodic and has the ***central block itinerary*** (J, X, J, S, J).
- This orbit transitions between the interior and exterior regions (the neighborhood of the 3:2 and 2:3 resonances, in particular). We call this kind of itinerary a ***resonance transition block***.

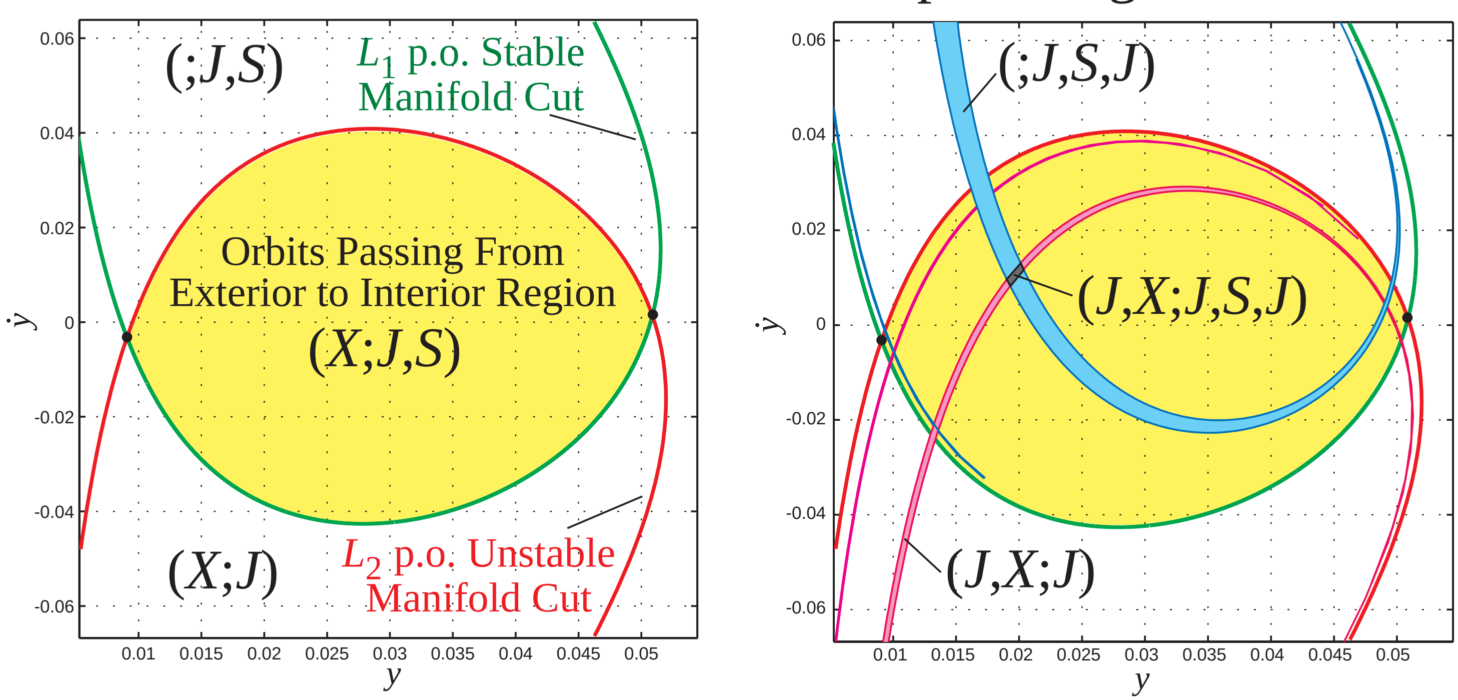
Poincare Section in the Jupiter Region



Rapid Transition Mechanism: Resonance Transition

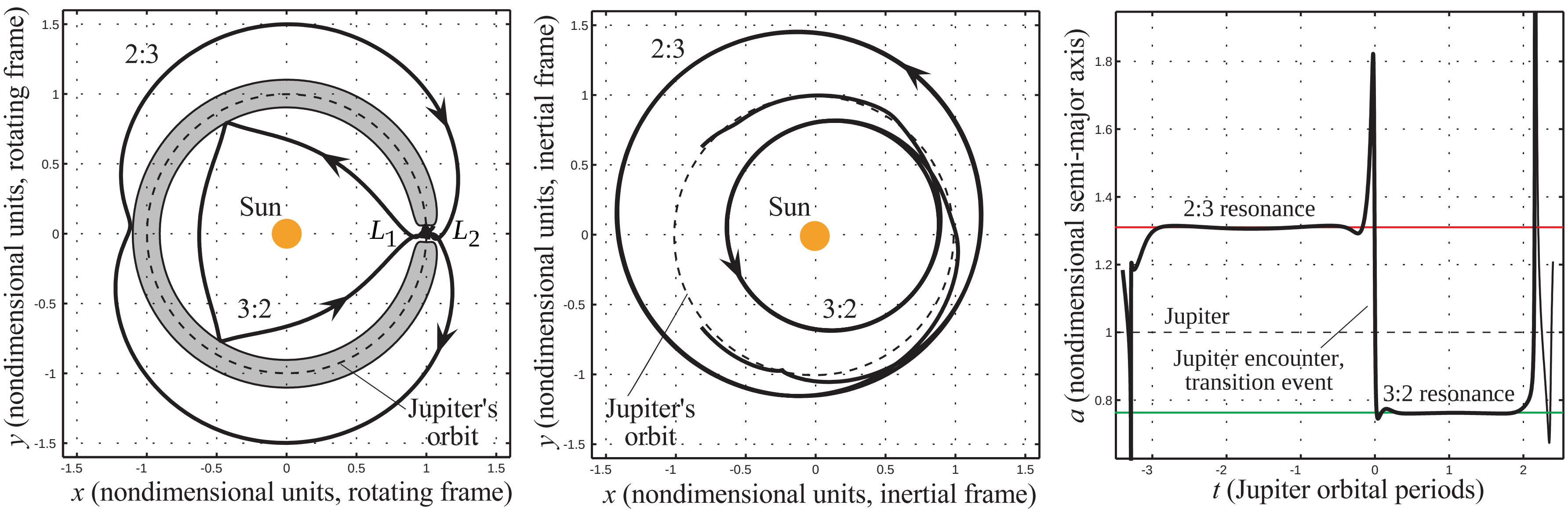
- This orbit makes a **rapid transition** from the exterior to the interior region and vice versa, passing through the Jupiter region. It will repeat this pattern **eternally**.
- While an orbit with this exact itinerary is very fragile, the structure of nearby orbits whose symbolic sequences have the same central block itinerary, namely (J, X, J, S, J), is quite ***robust***.

Poincare Section in the Jupiter Region



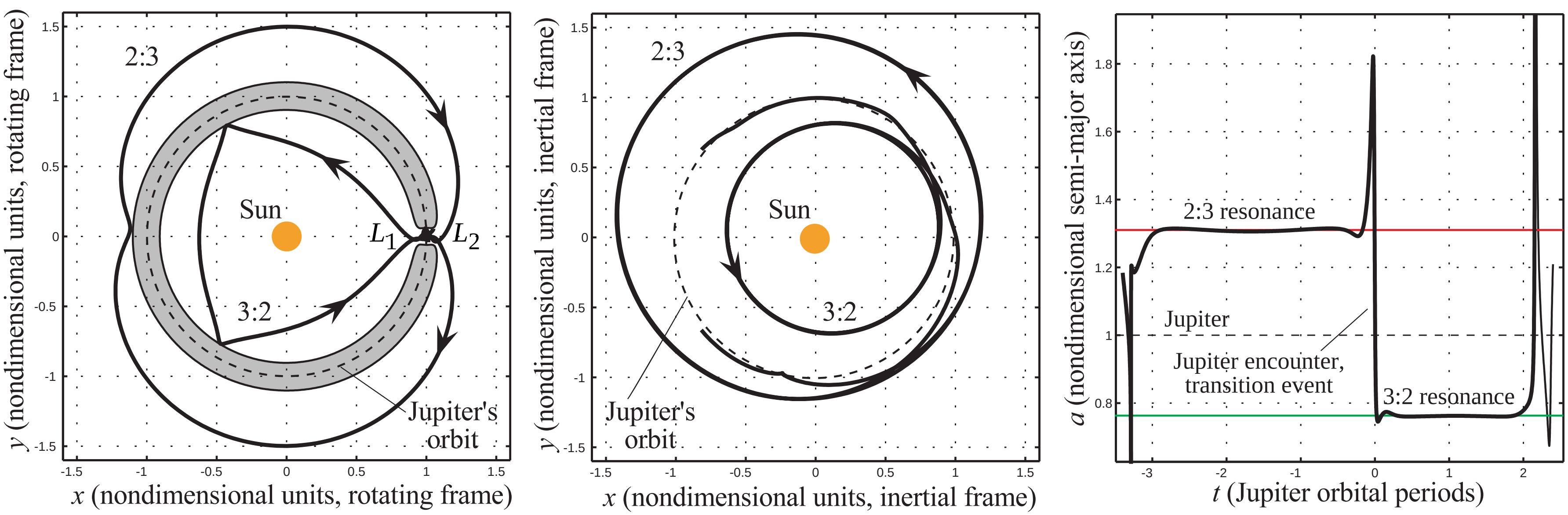
Rapid Transition Mechanism: Resonance Transition

- An example orbit with central block (J, X, J, S, J) is shown below.
- We will study how the dynamical channels near the chain connect the **3:2 resonance** of the interior region with the **2:3 resonance** of the exterior region.



PCR3BP: Perturbation of the Two-Body Problem

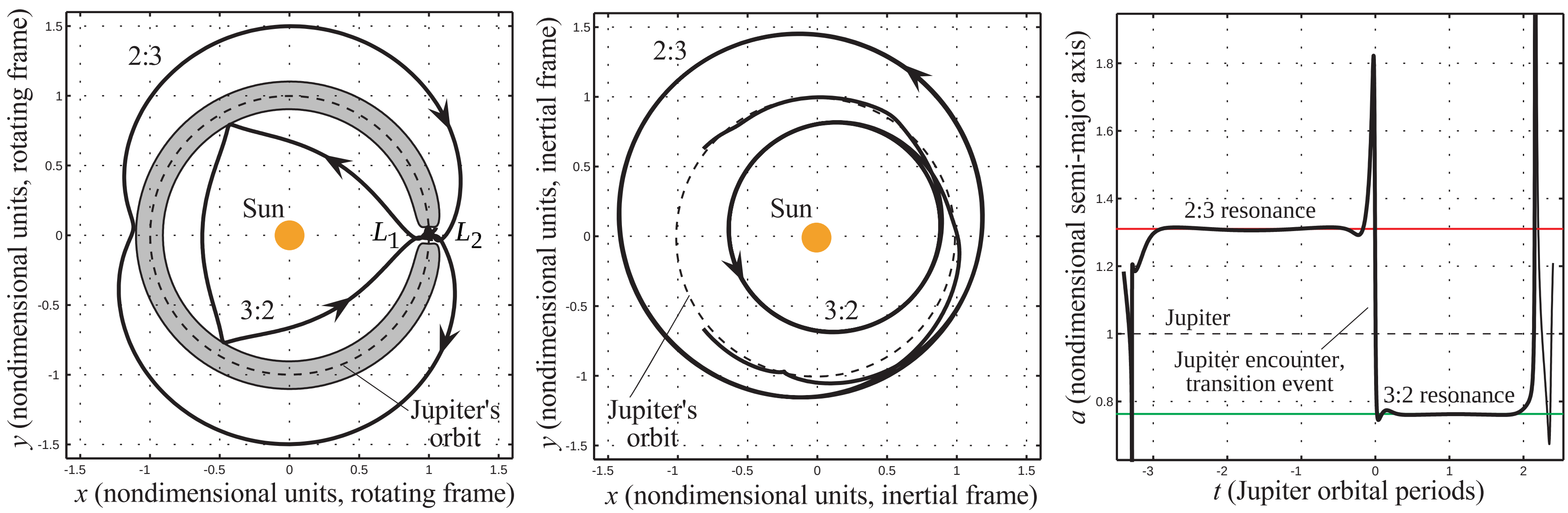
- Recall that the PCR3BP is a ***perturbation*** of the two-body problem. Hence, outside of a small neighborhood of L_1, the trajectory of a comet in the interior region follows essentially a ***two-body orbit*** around the Sun.
- In the heliocentric inertial frame, the orbit is nearly ***elliptical***.



Heliocentric Orbits: Mean Motion Resonance

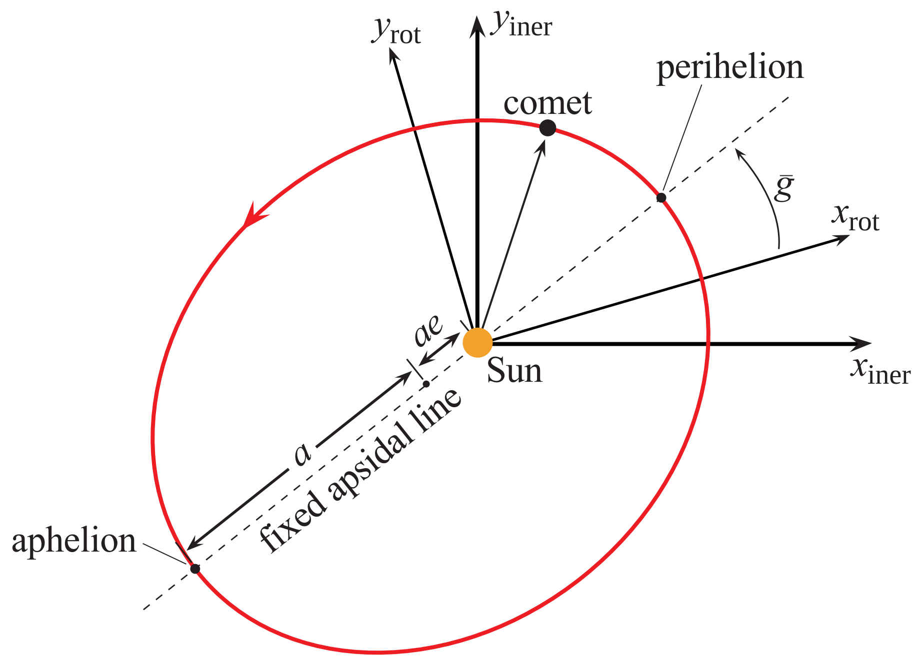
- The ***mean motion resonance*** of the comet with respect to Jupiter is equal to $a^{-3/2}$ where a is the semi-major axis of the heliocentric elliptical orbit. Recall that the Sun-Jupiter distance is normalized to be 1 in the PCR3BP.

- The comet is said to be in ***p:q resonance with Jupiter*** if $a^{-3/2} \approx p/q$, where p and q are small integers. In the heliocentric inertial frame, the comet makes roughly p ***revolutions*** around the Sun in q ***Jupiter periods***.



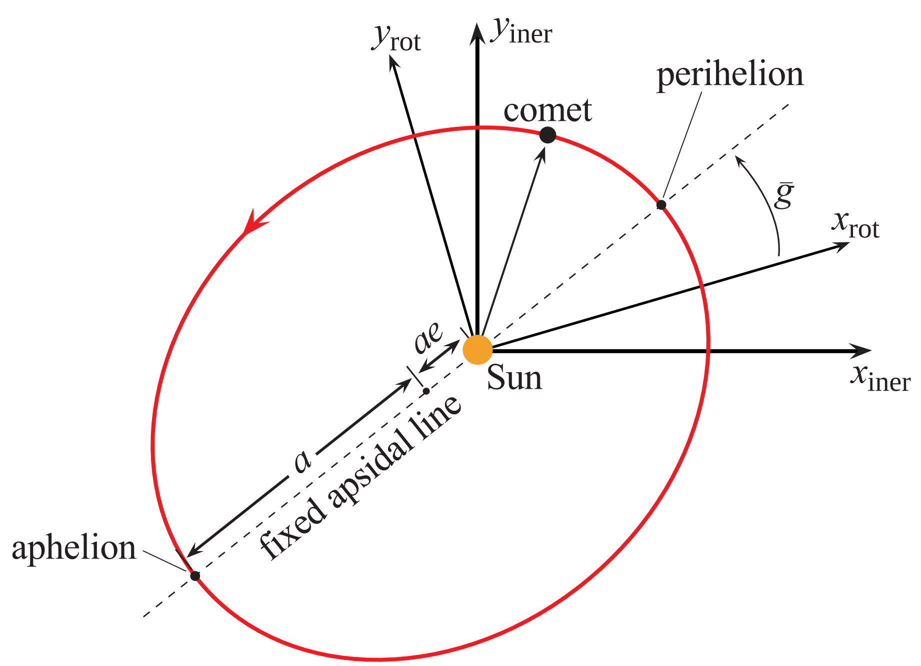
Canonical Coordinates: Delaunay Variables

- To study the process of resonance transition, we shall use a set of canonical coordinates, called ***Delaunay variables***, which make the study of the ***two-body regime*** of motion particularly simple.
- Delaunay variables in the rotating coordinates are denoted $(l, \bar{g}, L, G)$. $G = [a(1-e^2)]^{1/2}$ is the angular momentum. L is related to the semi-major axis a, by $L = a^{1/2}$, hence encodes the ***mean motion resonance*** (with respect to Jupiter in the Sun-Jupiter system).

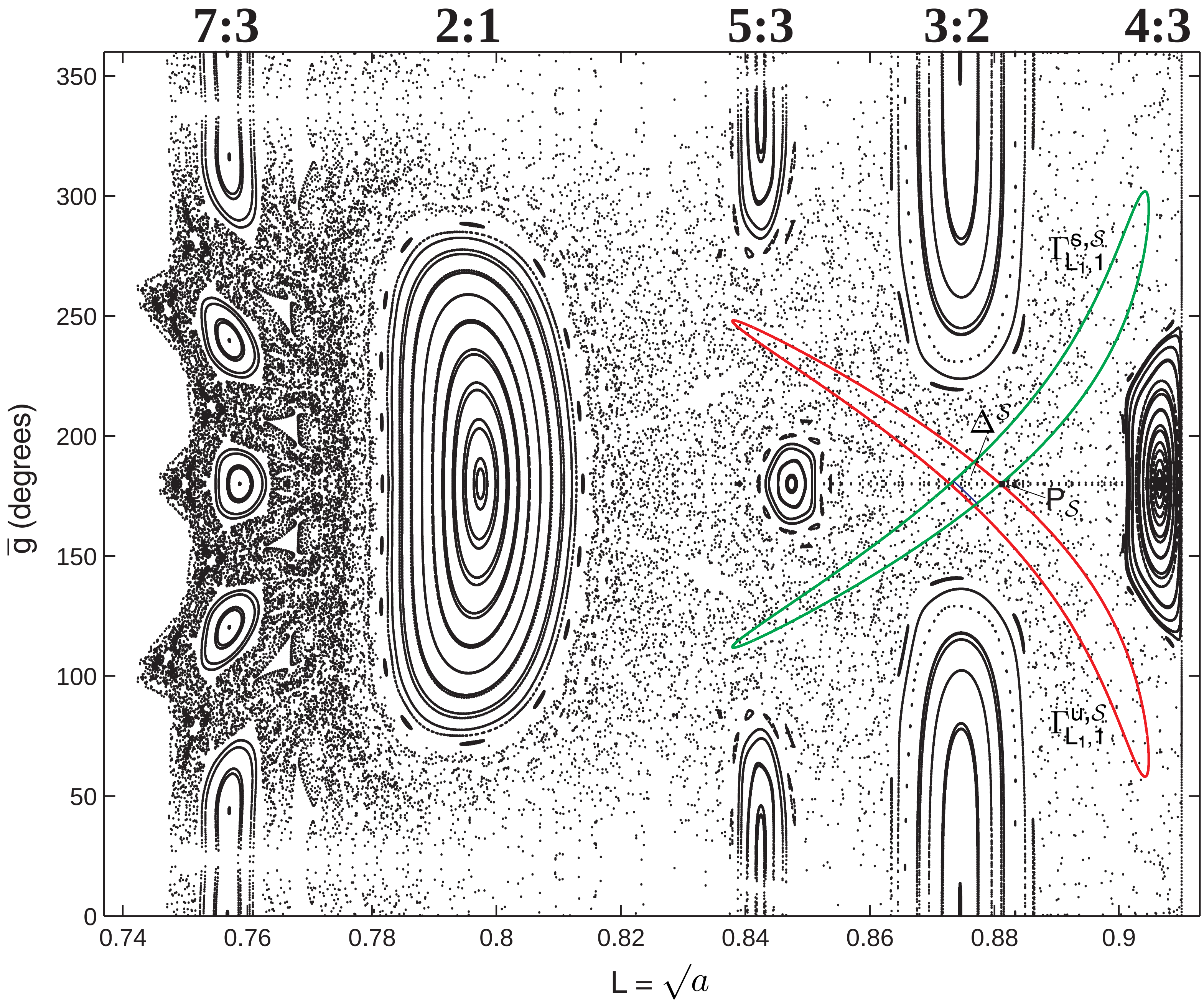


Canonical Coordinates: Delaunay Variables

- Both l and $\bar{g}$ are angular variables defined modulo 2π.
- $\bar{g}$ is the ***argument of perihelion*** relative to the rotating axis.
- l is the ***mean anomaly***, the ratio of the area swept out by the ray from the Sun to the comet starting from its perihelion passage to the total area.
- Szebehely [1967], Abraham & Marsden [1978], Meyer & Hall [1992].



Interior Resonances: U_1 Poincaré Section $(L,\bar{g})$



Interior Resonances: U_1 Poincaré Section

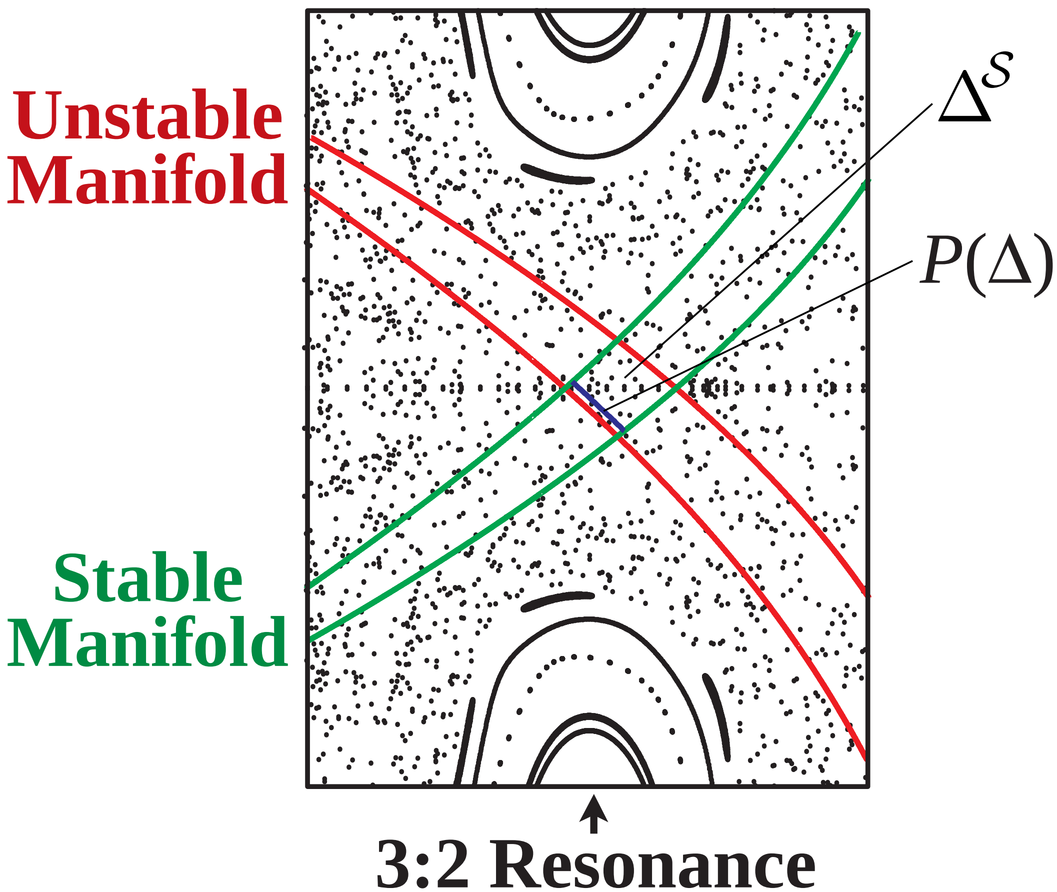
- The striking thing is that the first cuts of the **stable** and **unstable** manifolds ***intersect*** exactly at the region of the ***3:2 resonance***.
- Their intersection $\Delta^{\mathcal{S}}$ contains all the orbits that have come from the Jupiter region J into the interior region S, gone around the Sun once (in the rotating frame), and will return to the Jupiter region. In the heliocentric inertial frame, these orbits are nearly elliptical outside a neighborhood of L_1.
- They have a semi-major axis which corresponds to 3:2 resonance by Kepler's law (i.e., $a^{-3/2} = L^{-3} \approx 3/2$). Therefore, ***any*** Jupiter comet which has an ***energy similar to Oterma's*** and which circles around the Sun ***once*** in the interior region ***must be in 3:2 resonance*** with Jupiter.

Mixed Phase Space: Stable "Islands" & Chaotic "Sea"

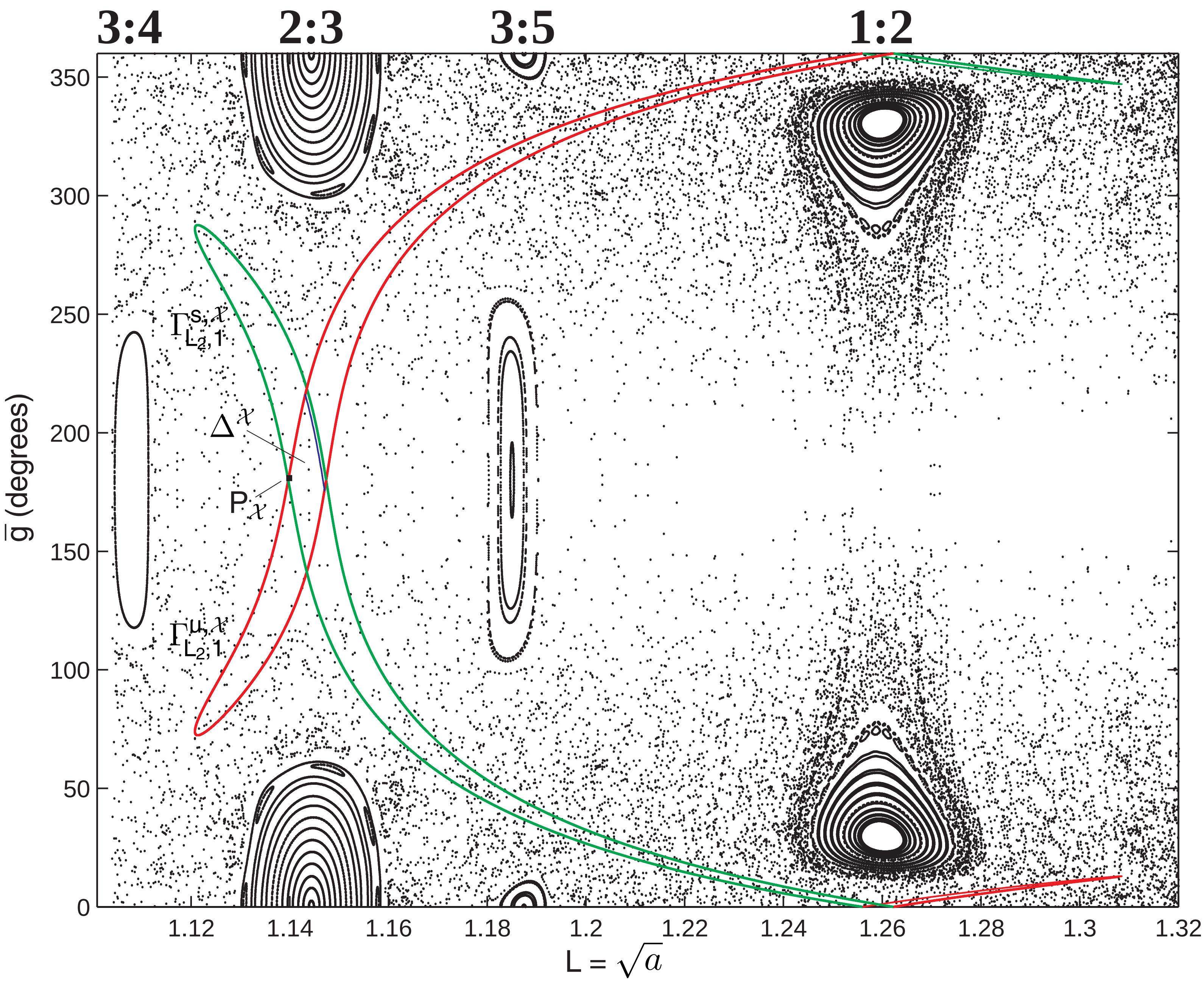
- The black background points on the U_1 Poincaré section reveal the character of the interior region phase space for this energy surface.
- ***Mixed phase space*** of stable periodic and quasiperiodic ***invariant tori "islands"*** embedded in bounded ***chaotic "sea."***
- The ***families of stable tori***, where a "family" denotes those tori islands which lie along a strip of nearly constant L, correspond to ***mean motion resonances***. The size of the tori island corresponds to the dynamical significance of the resonance. The number of tori islands equals the order of the resonance (e.g., 3:2 is order 1, 5:3 is order 2).
- In the center of each island, there is a point corresponding to an exactly periodic, stable, resonant orbit. In between the stable islands of a particular resonance (i.e., along a strip of nearly constant L), there is a saddle point corresponding to an exactly periodic, unstable, resonant orbit. In the figure, the manifold intersection region $\Delta^{\mathcal{S}}$ is centered on this saddle point for the 3:2 resonance.

Connection Between Interior and Exterior Resonances

- A subset of the interior resonance intersection region $\Delta^{\mathcal{S}}$ is connected to exterior resonances through a heteroclinic intersection in the Jupiter region. This small ***blue strip*** inside $\Delta^{\mathcal{S}}$ is part of the dynamical channel connecting interior and exterior resonances, and is thus the ***resonance transition mechanism*** we seek.



Exterior Resonances: U_4 Poincaré Section $(L,\bar{g})$



Exterior Resonances: U_4 Poincaré Section $(L,\bar{g})$

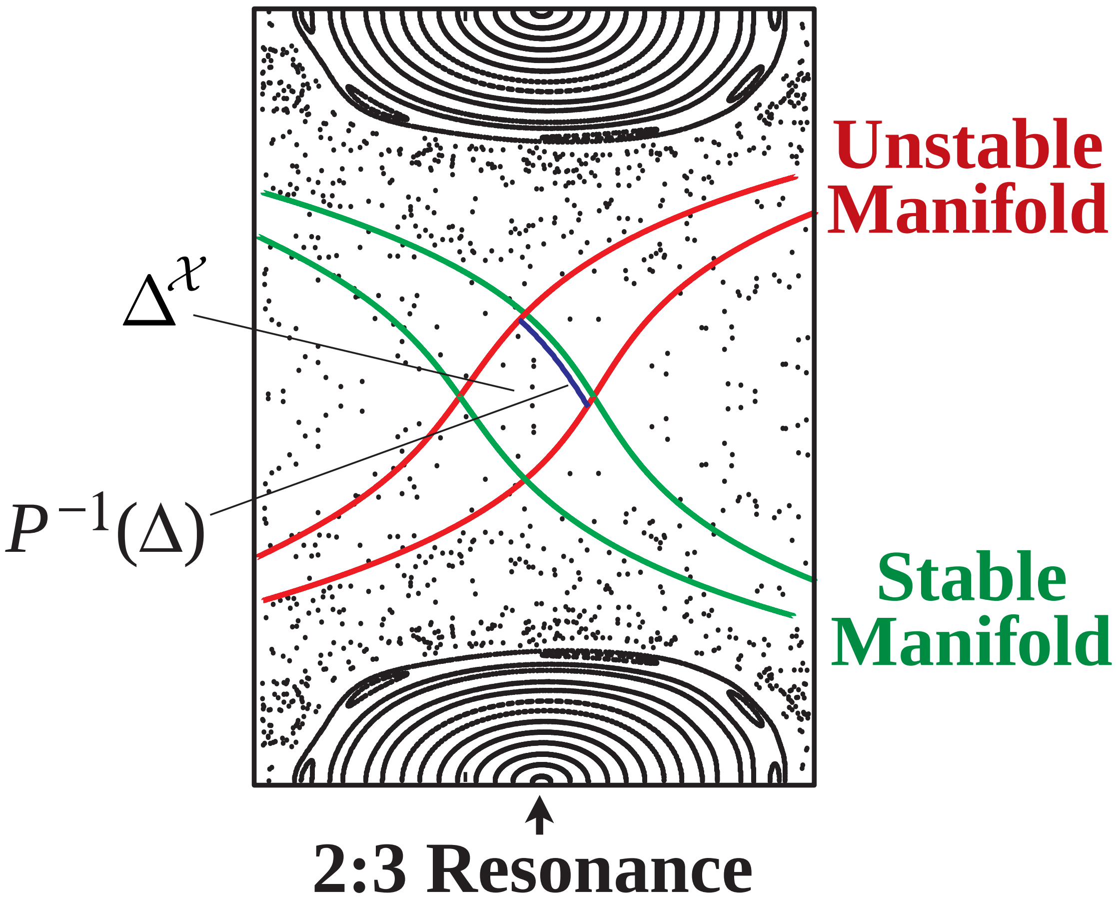
- We show the first exterior region Poincaré cuts of the **stable** and **unstable** manifolds of an L_2 periodic orbit with the U_4 section on the same energy surface.
- Notice that the first cuts of the **stable** and **unstable** manifolds intersect at ***two places***; one of the intersections is exactly at the region of the ***2:3 resonance***, the other is at the ***1:2 resonance***.
- The intersection $\Delta^{\mathcal{X}}$ contains all the orbits that have come from the Jupiter region J into the exterior region X, have gone around the Sun once (in the rotating frame), and will return to the Jupiter region. Note that $\Delta^{\mathcal{X}}$ has two components (the 2:3 and 1:2 resonance regions).

Exterior Resonances: U_4 Poincaré Section $(L, \bar{g})$

- In the heliocentric inertial frame, these orbits are nearly elliptical outside a neighborhood of L_2. They have a semi-major axis which corresponds to either 2:3 or 1:2 resonance by Kepler's law. Therefore, any Jupiter comet which has an energy similar to *Oterma*'s and which circles around the Sun once in the exterior region must be in either 2:3 or 1:2 resonance with Jupiter.
- The background points were generated by a technique similar to those in the interior resonance Poincaré section. They reveal a similar mixed phase space, but now the resonances are exterior resonances (exterior to the orbit of Jupiter). We see that the exterior resonance intersection region $\Delta^{\mathcal{X}}$ envelops both the 2:3 and the 1:2 unstable resonance points.

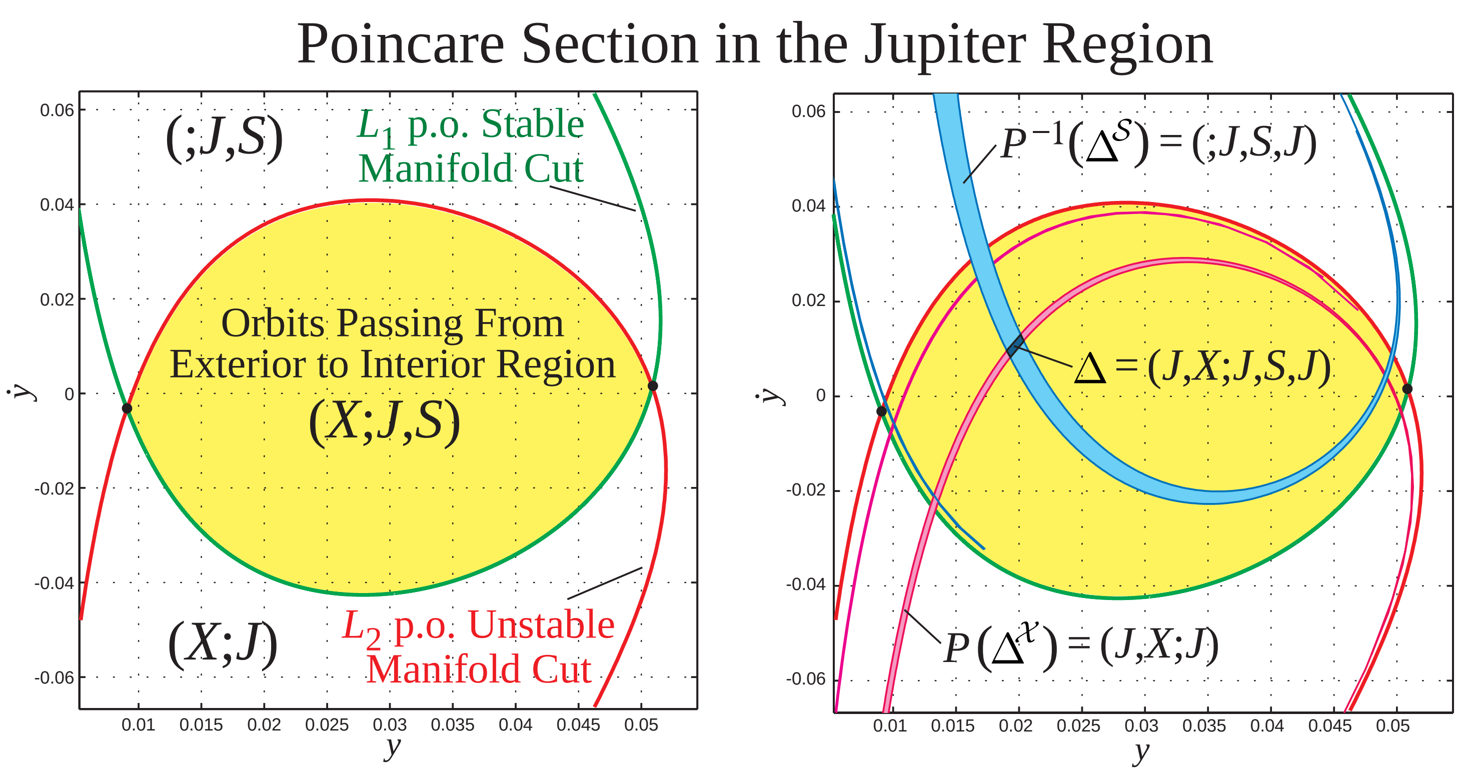
Connection Between Interior and Exterior Resonances

- A portion of $\Delta^{\mathcal{X}}$ is connected to interior resonances through a heteroclinic intersection in the Jupiter region. In particular, the small ***blue strip*** inside the ***2:3 intersection region*** connects to the ***3:2 intersection region*** of $\Delta^{\mathcal{S}}$ (and is its pre-image). We have thus found the ***resonance transition*** used by ***Oterma***.



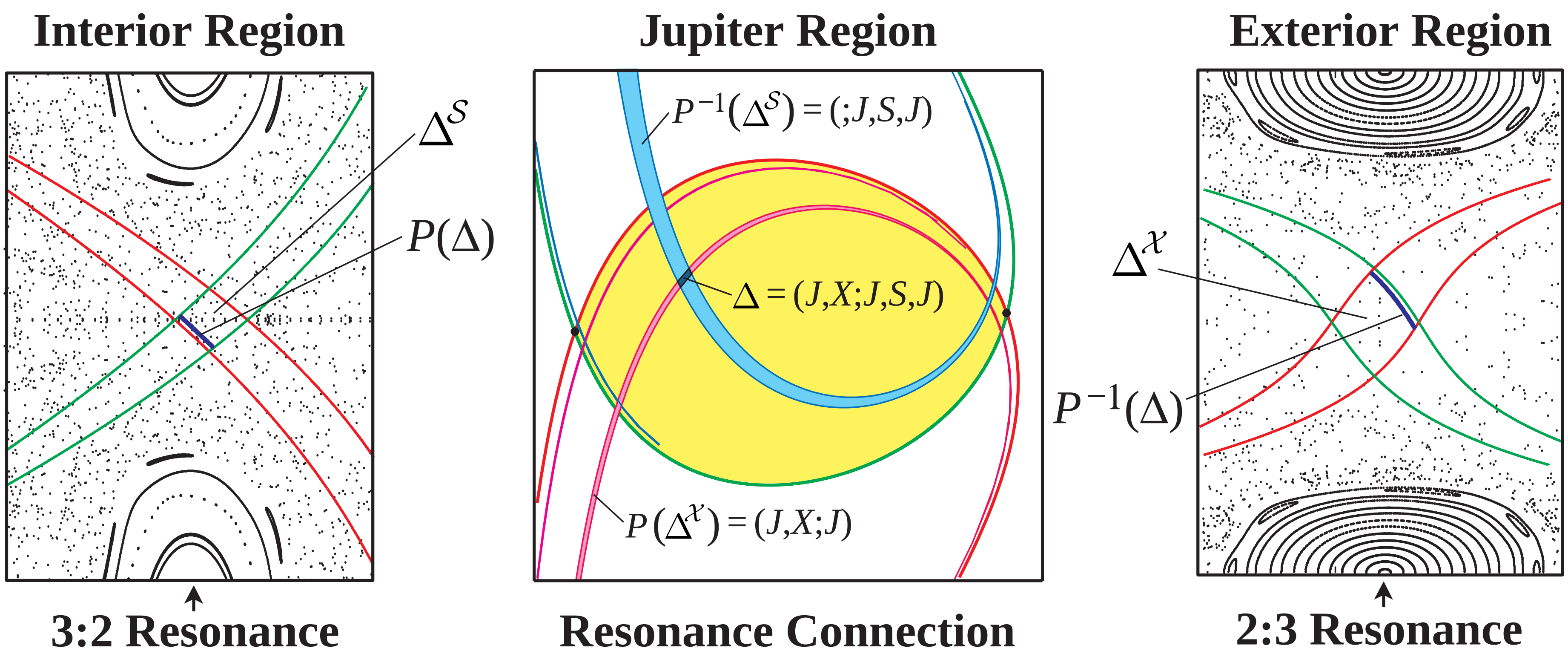
Connection Between Interior and Exterior Resonances

- We have referred to a heteroclinic intersection Δ connecting interior $\Delta^{\mathcal{S}}$ and exterior $\Delta^{\mathcal{X}}$ resonance intersection regions. Below, we show image of $\Delta^{\mathcal{X}}$ (2:3 resonance portion) and pre-image of $\Delta^{\mathcal{S}}$ in the J region. Their intersection $\Delta = P(\Delta^{\mathcal{X}}) \cap P^{-1}(\Delta^{\mathcal{S}})$ contains all orbits whose itineraries have central block $(J, X; J, S, J)$, corresponding to at least one transition between the exterior 2:3 resonance and interior 3:2 resonance.



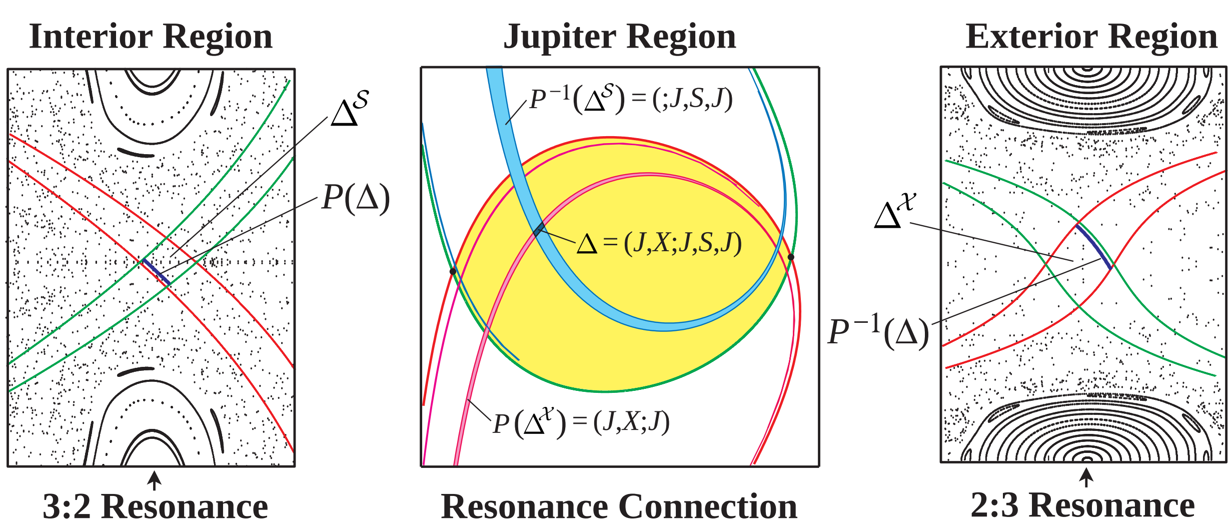
Connection Between Interior and Exterior Resonances

- Δ contains orbits in ***transition*** between the 2:3 to 3:2 resonances.
- Comets such as ***Oterma*** have passed through analogous regions.



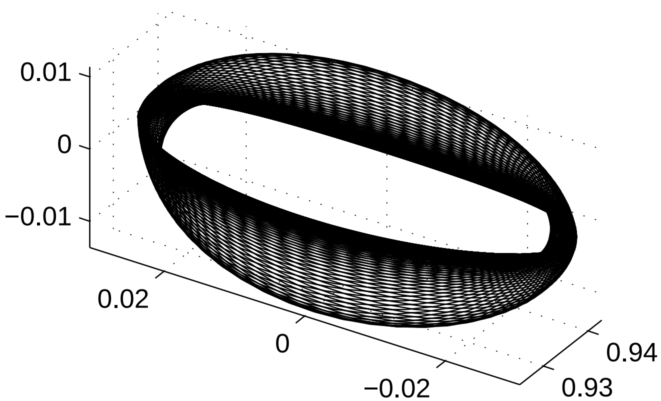
Resonance Connection for Three Degrees of Freedom

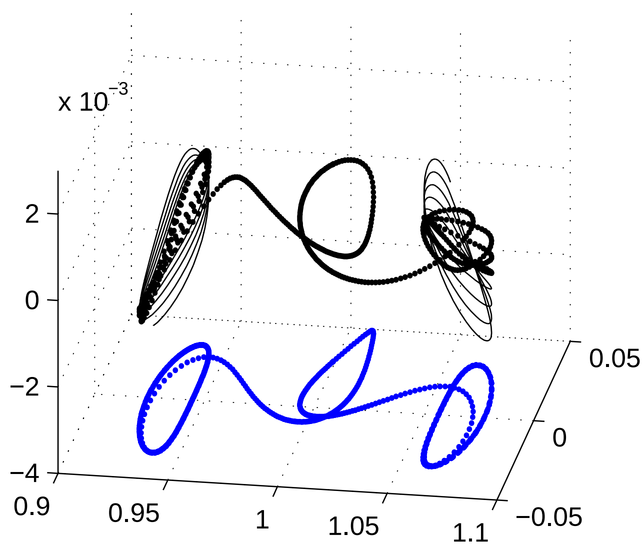
- It is reasonable to conclude that, within the full three-dimensional model, ***Oterma***'s orbit lies in an analogous region of phase space.
- It is therefore within the L_1 and L_2 periodic and quasiperiodic orbit manifold ***tubes***, whose complex global dynamics lead to ***intermittent behavior***, including ***resonance transition***.
- More study is needed for a thorough understanding of the resonance transition phenomenon. The tools developed in this course (***dynamical channels, symbolic dynamics***, etc.) should lay a firm theoretical foundation for any such future studies.



Future Work: Extension to Three Dimensions

- Natural extension: apply same methodology to ***3D CR3BP***.
  - Seek homoclinic & heteroclinic orbits associated with 3D periodic "halo" & quasi-periodic "quasi-halo" & Lissajous orbits about L_1 & L_2. Dimension count suggests heteroclinic intersections exist.
  - Union would be 3D homoclinic-heteroclinic chains around which symbolic dynamics could be used to track a variety of exotic orbits.
  - ***Three-dimensional dynamical channels*** will provide more complete understanding of phase space transport mechanisms.



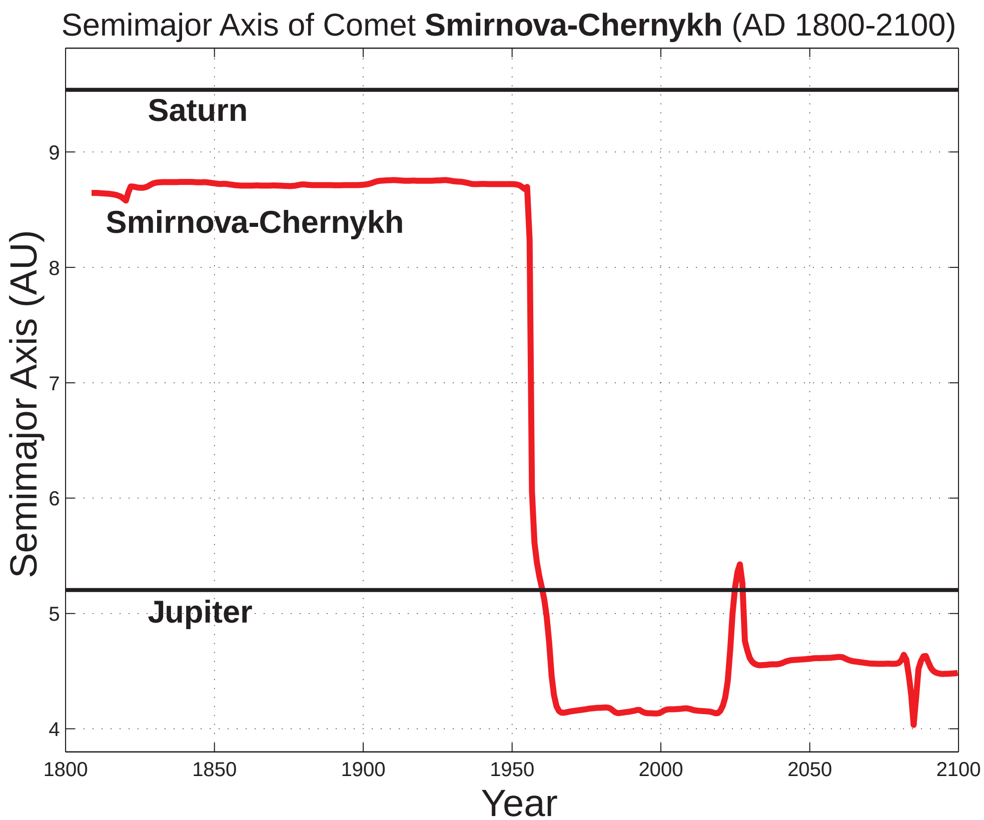


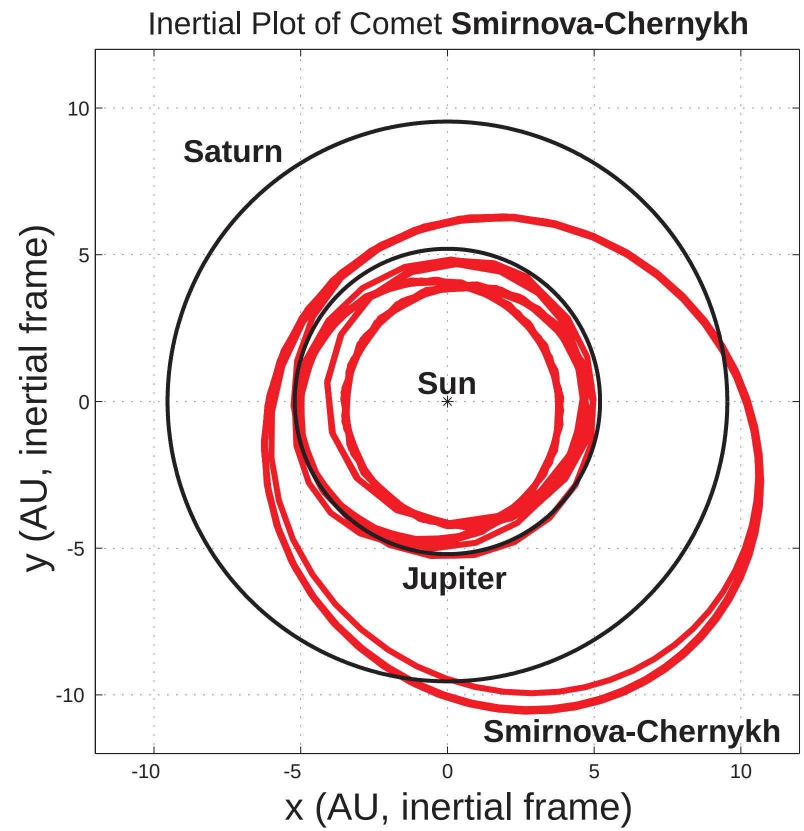
Future Work: Coupling of Two 3-Body Systems

- Dynamics governing ***transport between adjacent planets***.
  - Coupled 3-body problem: e.g., comet between Jupiter & Saturn.
  - Between the two planets, the comet's motion is mostly heliocentric, but is precariously poised between two competing three-body dynamics.
  - In this region, heteroclinic orbits connecting Lyapunov orbits of the two different three-body systems may exist, leading to complicated transfer dynamics between the two adjacent planets.

Comet Transition Between Jupiter and Saturn

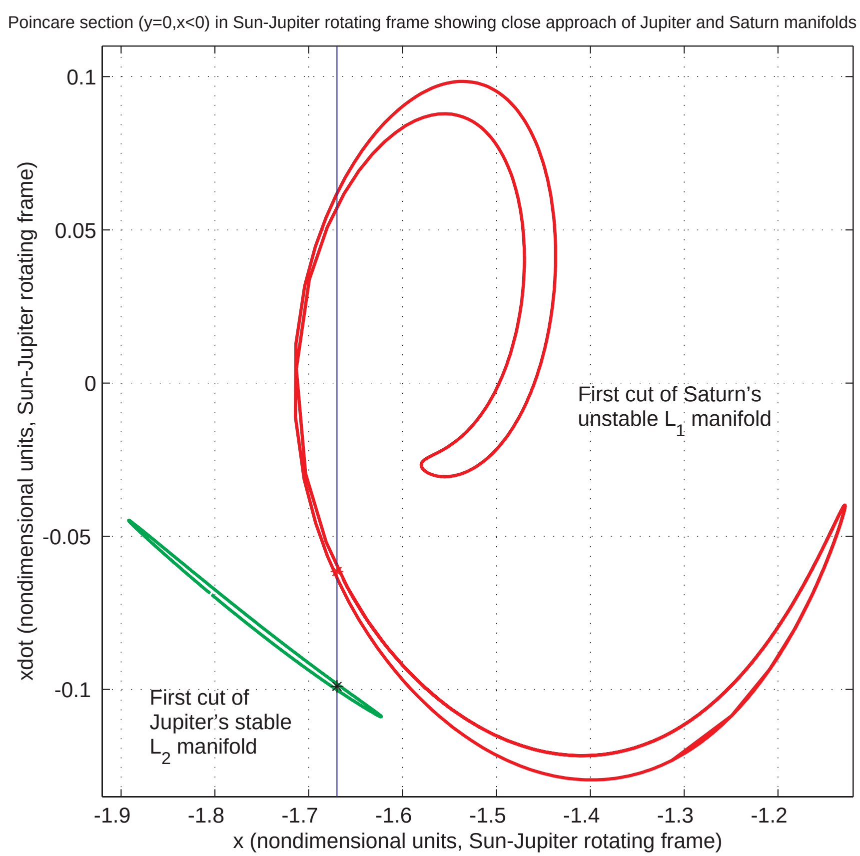
- Example: Comet ***Smirnova-Chernykh*** undergoes a **rapid transition** from **Saturn**'s control to **Jupiter**'s control.

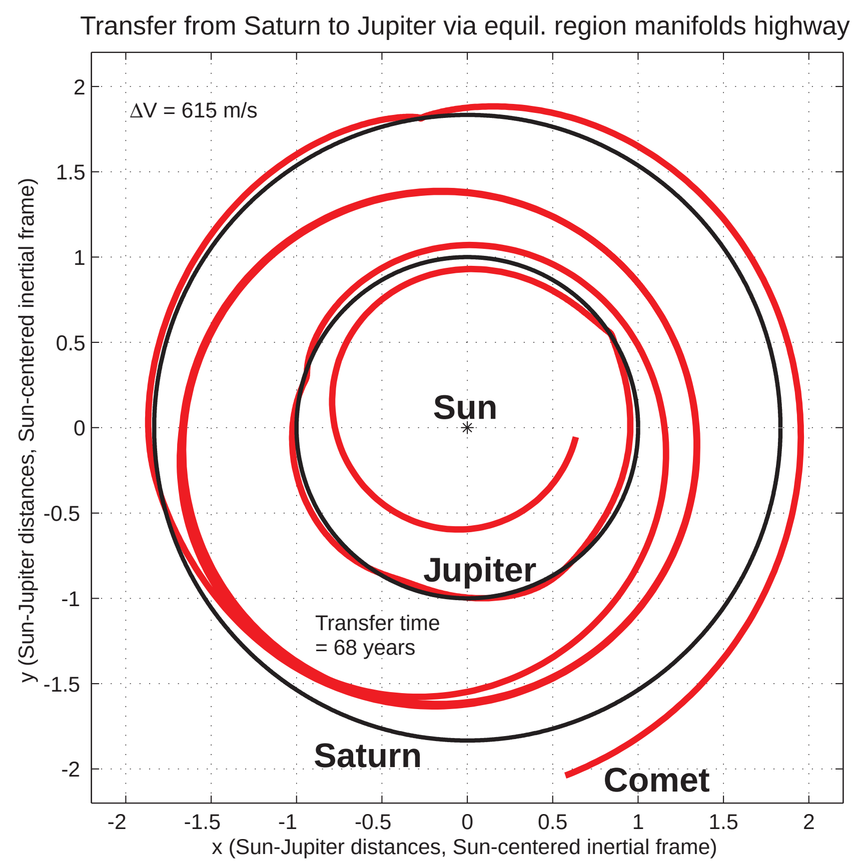




Comet Transition Between Jupiter and Saturn

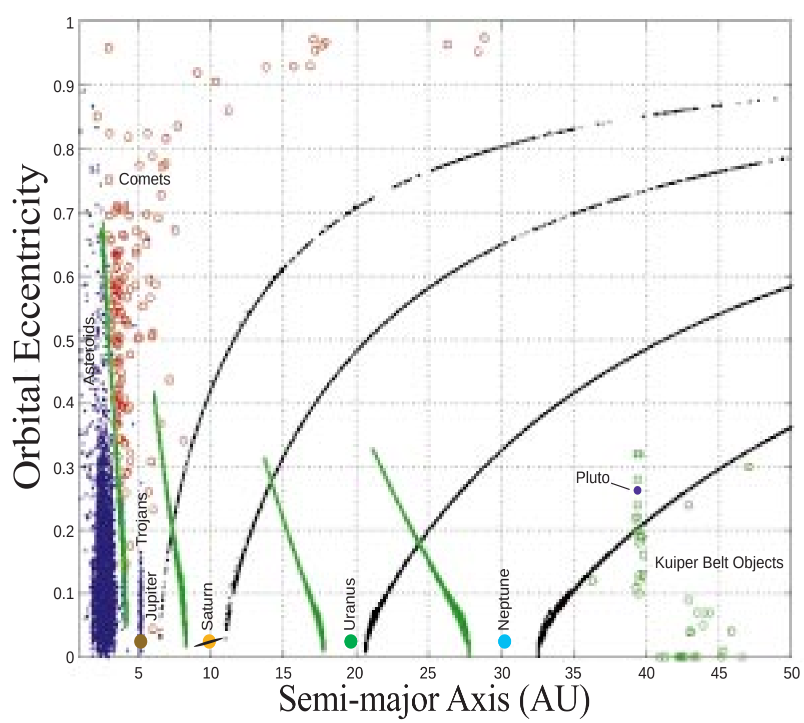
- Coupled PCR3BP shows near ***intersections*** between Lyapunov orbit manifold tubes of Jupiter and Saturn (requiring mild ΔV).
  - Natural continuous thrust of comet ***outgassing*** may be enough.
  - Longer time integration will likely reveal genuine intersections.





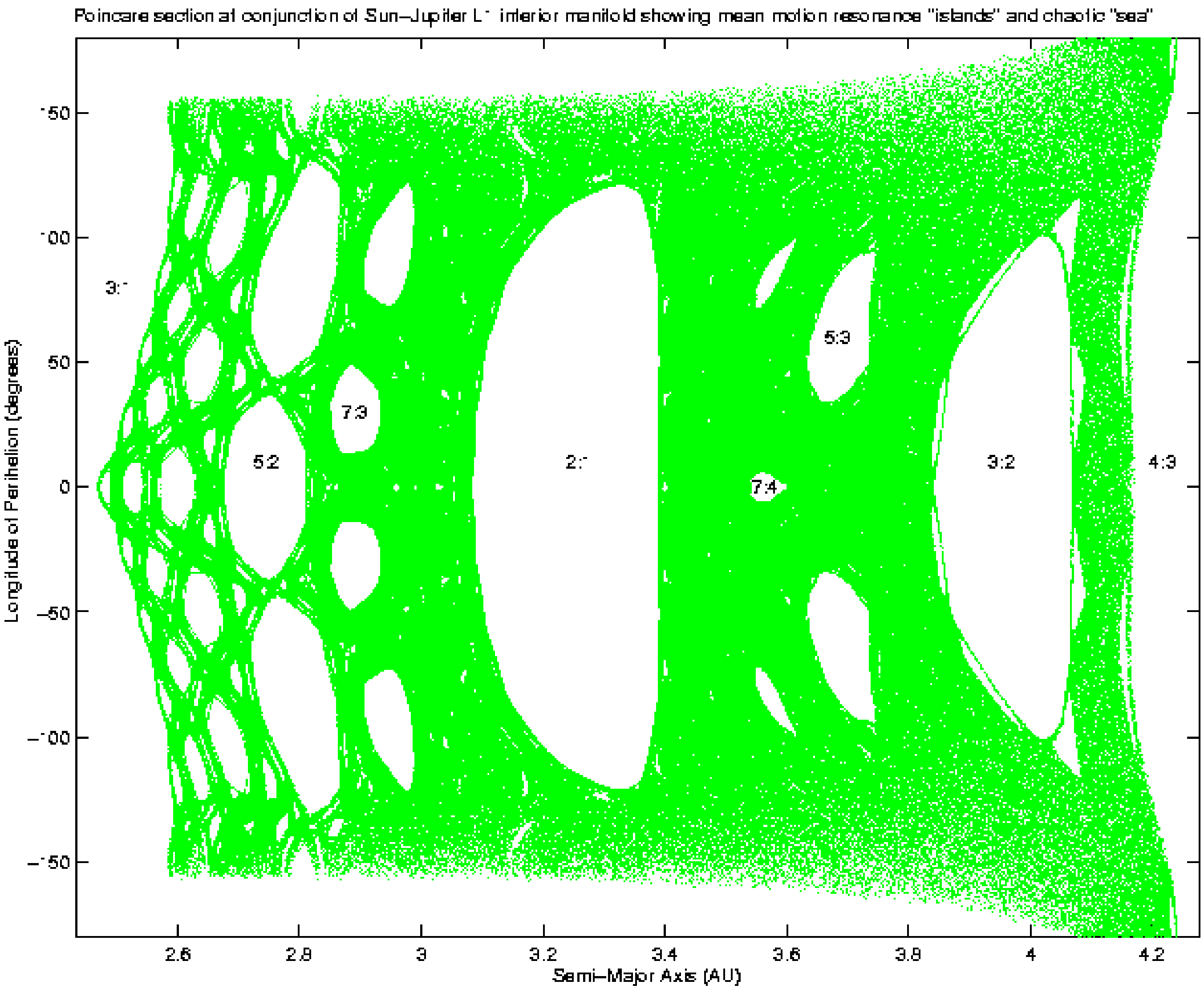
Future Work: Long Time Integration

- Results limited thus far to short time (a few periods of Jupiter).
  - **Long time** integration (millions of Jupiter periods) will reveal ***statistical*** information and ***new phenomena***.
  - Preliminary results suggest manifold structures associated to L_1 and L_2 have helped sculpt the ***solar system*** and ***transport material*** between the planets.



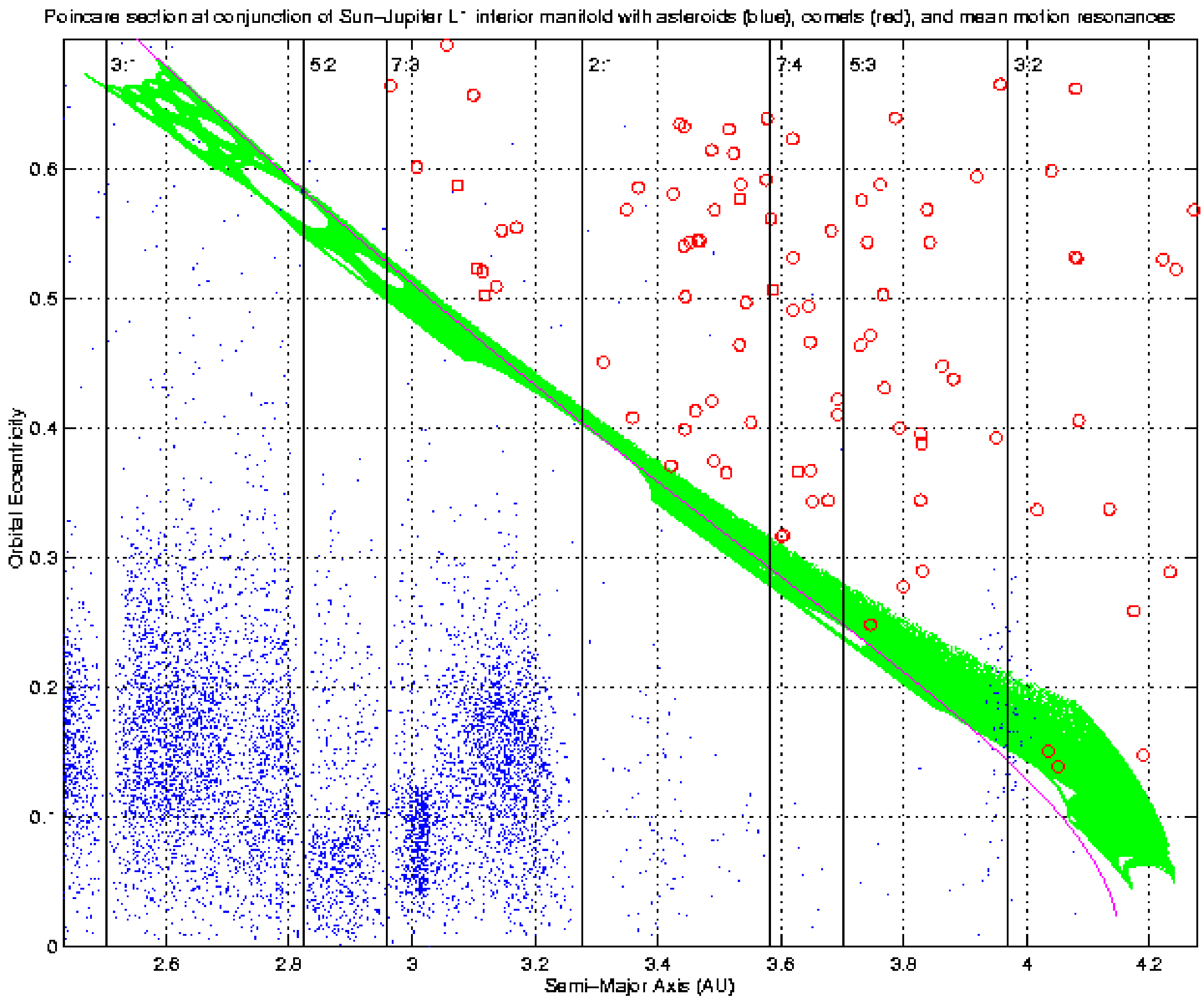
Long Time Integration: Jupiter's L_1 Manifolds

- We show U_1 Poincaré section of Jupiter's L_1 stable & unstable manifolds for one million iterations (in a vs. $\bar{g}$).



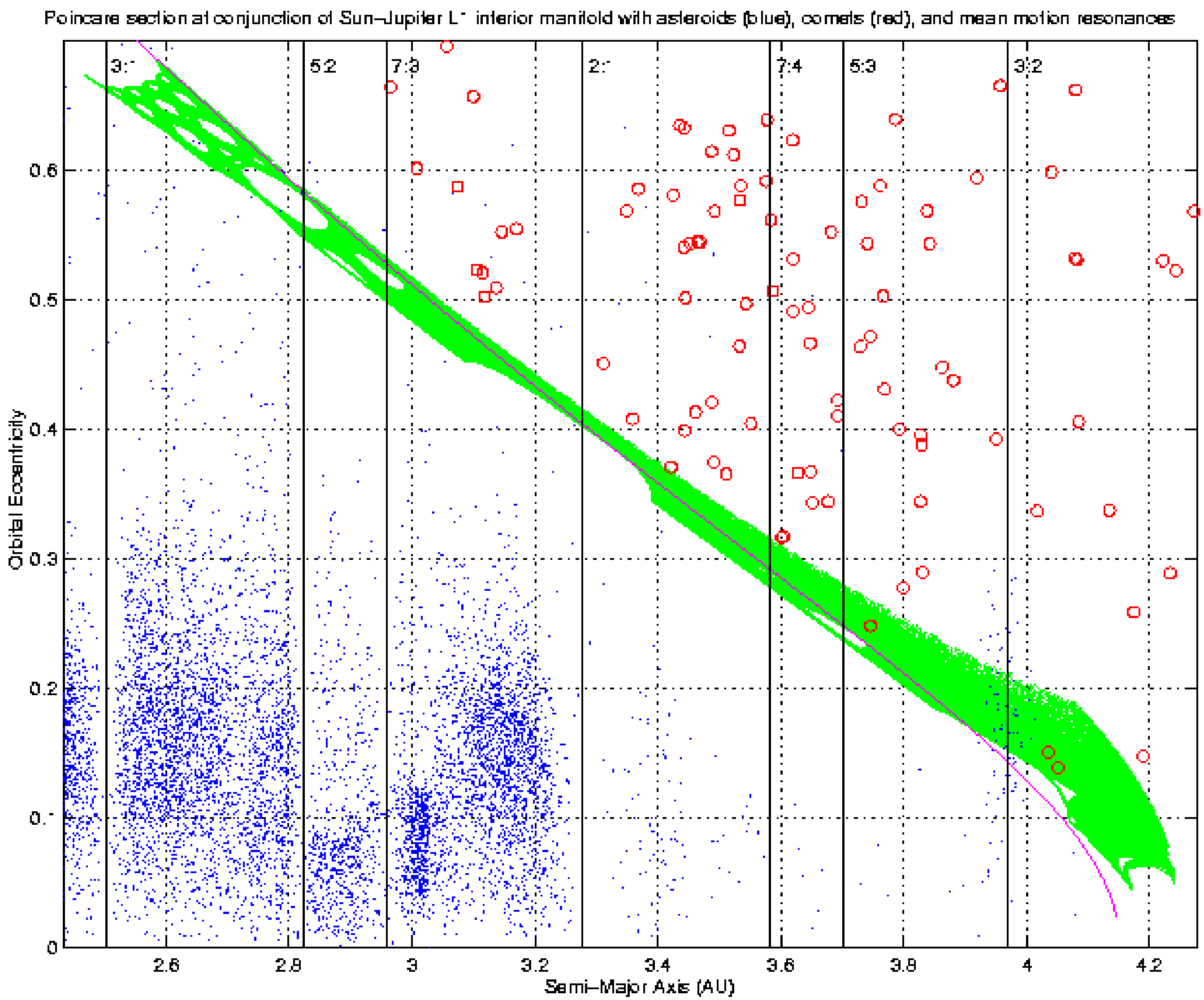
Long Time Integration: Jupiter's L_1 Manifolds

- We can also plot this in semimajor axis a vs. eccentricity e. Away from L_1, manifold hugs curve given by $C = \frac{1}{a} + 2\sqrt{a(1-e^2)}$.



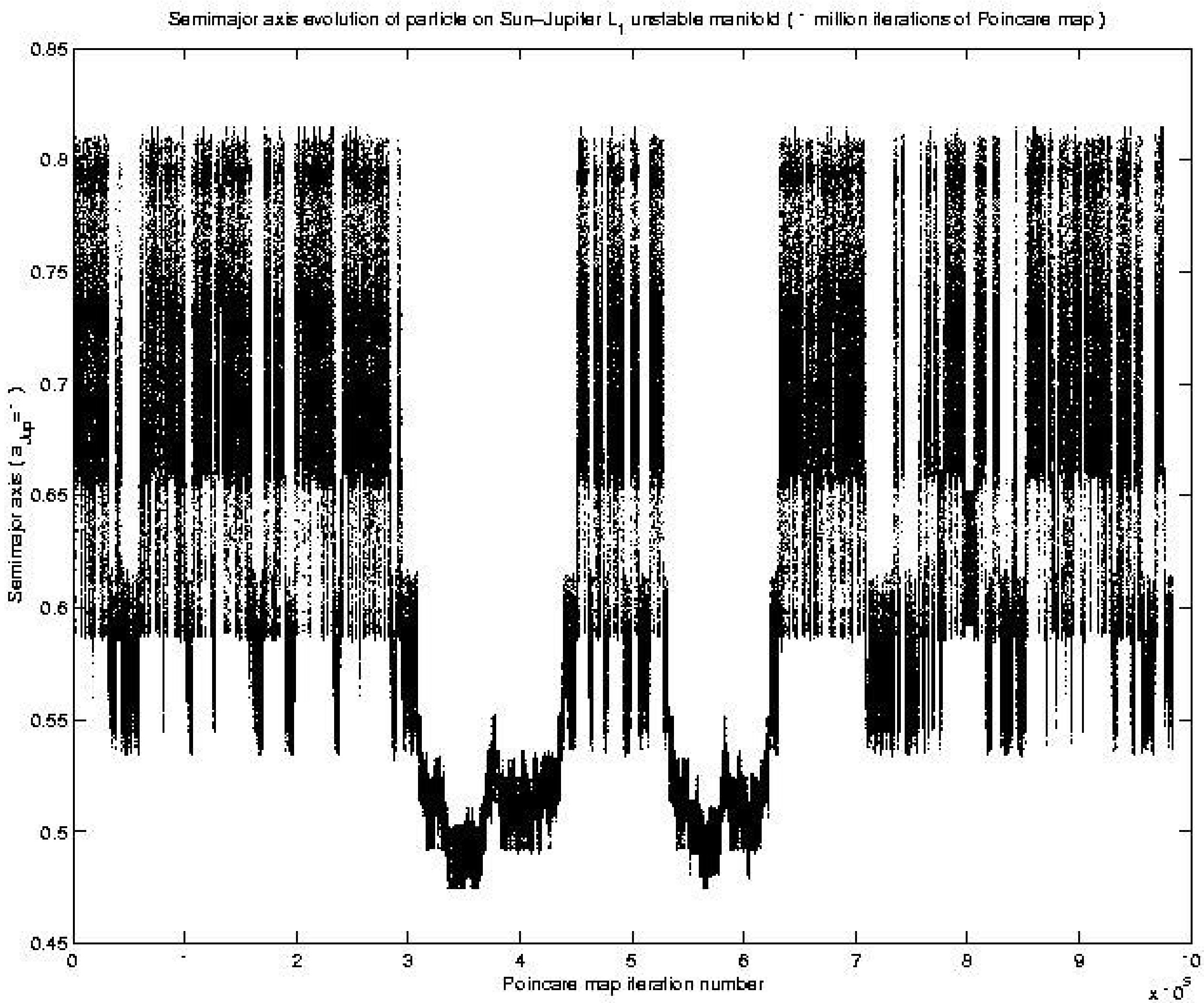
Long Time Integration: Jupiter's L_1 Manifolds

- Note how ***manifold*** acts as stability boundary, separating stable ***asteriods*** from unstable ***comets***.



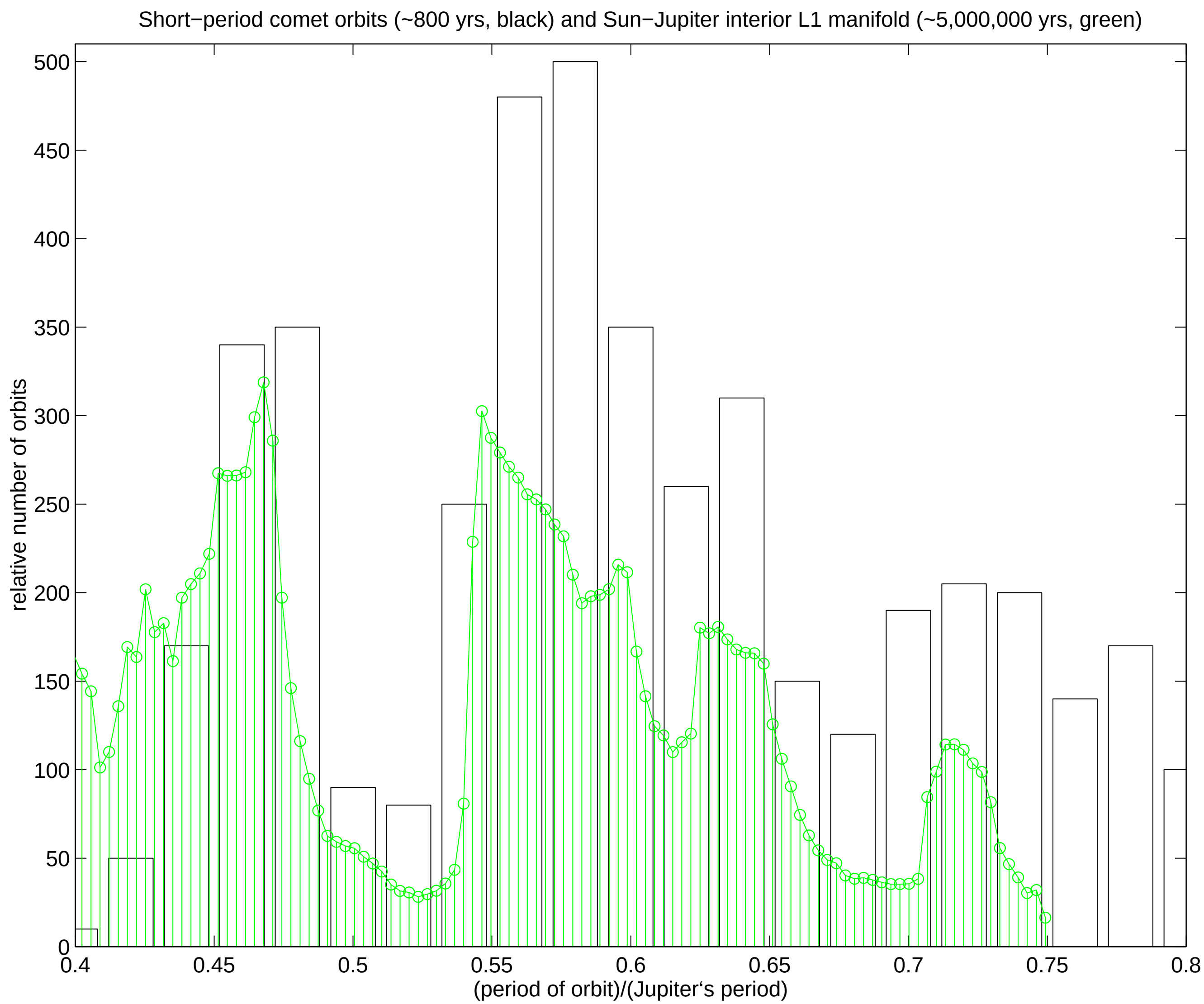
Intermittent Behavior Along Jupiter's L_1 Manifolds

- Time history of semimajor axis a for one million iterations shown.
- Manifold exhibits ***intermittency***, ***jumping***, ***sticking***.



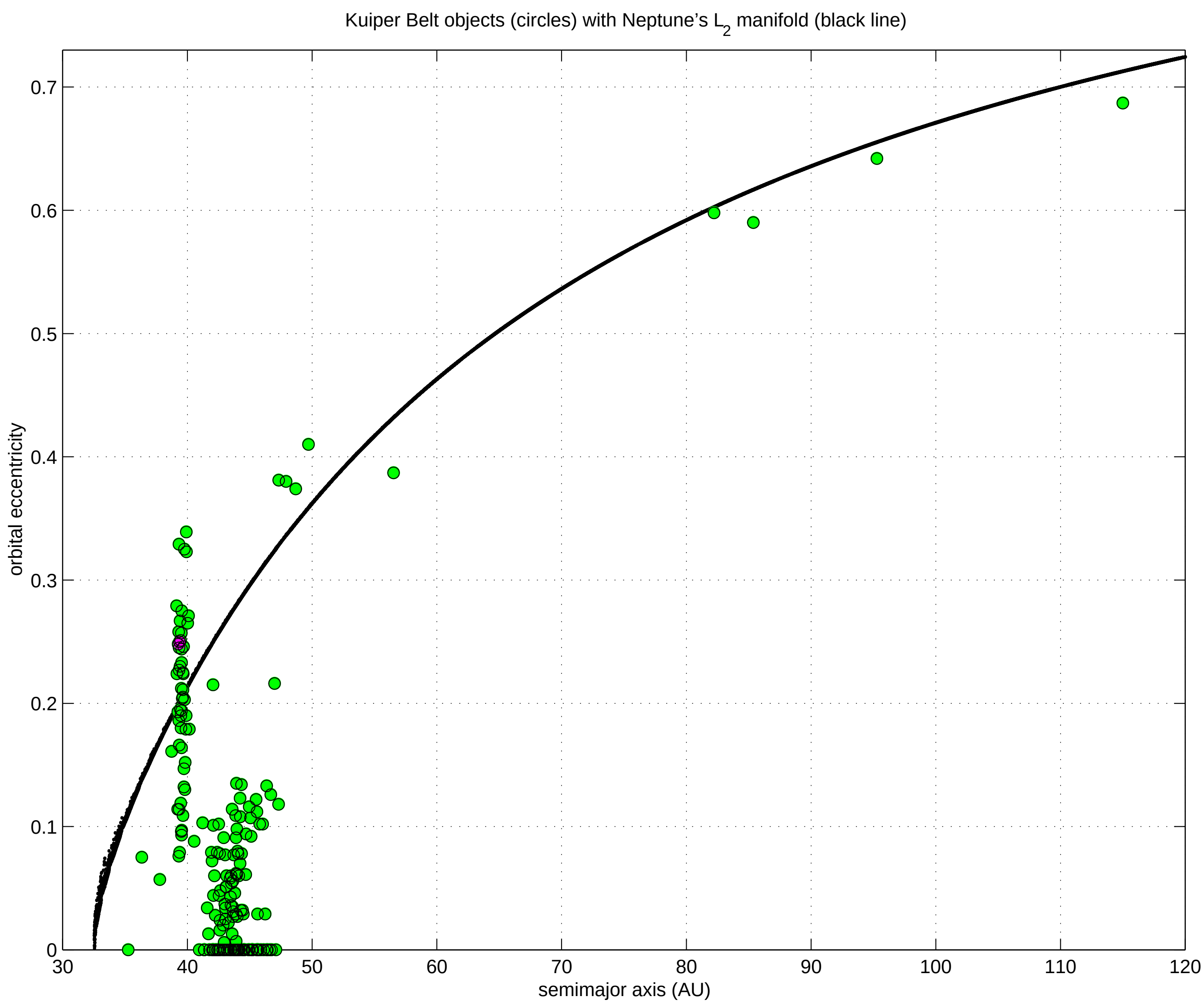
Comet Distribution Matches Jupiter's L_1 Manifolds

- Taking histogram of Jupiter's L_1 manifolds shows fair agreement with distribution of Jupiter comets. Same dynamics is at work.



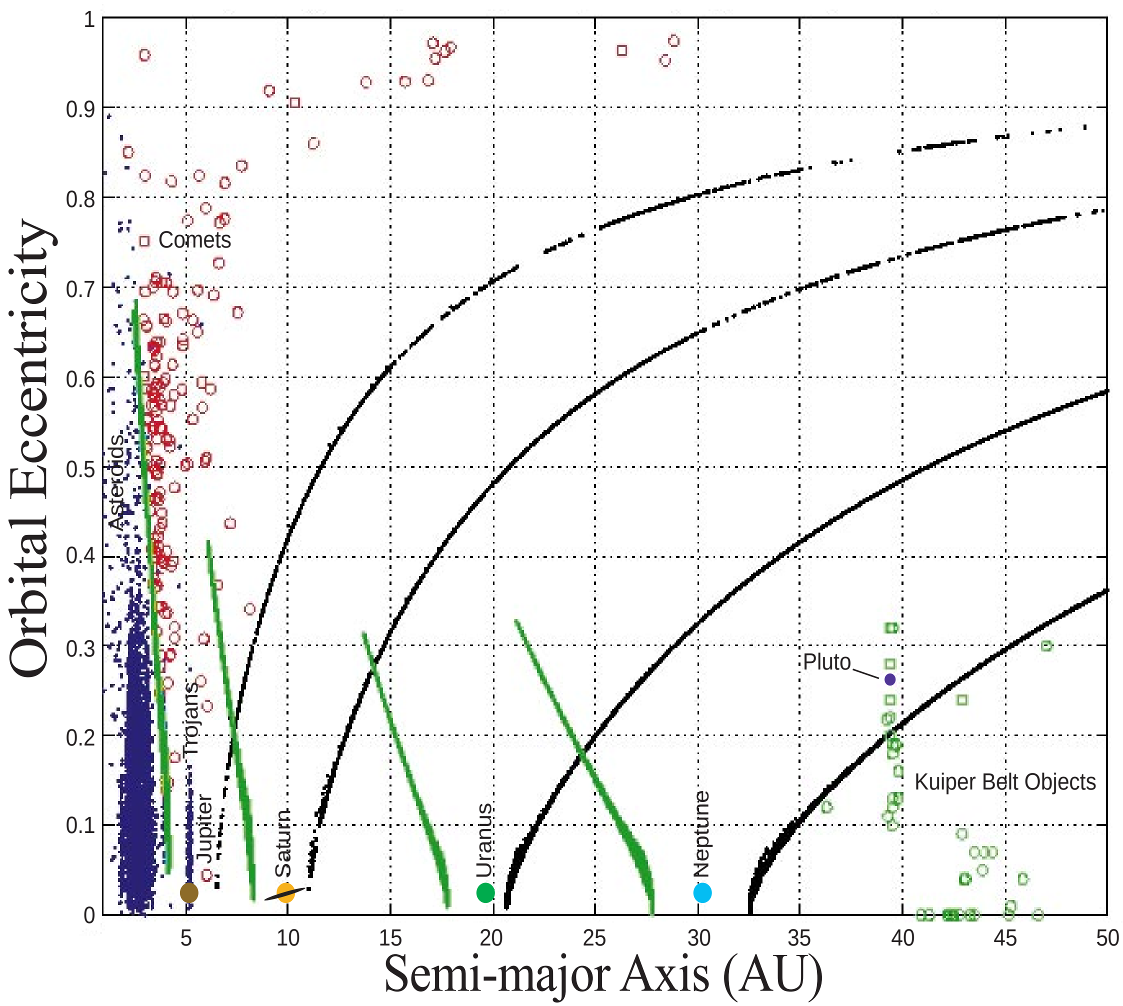
Kuiper Belt and Neptune's L_2 Manifolds

- Just as Jupiter's manifolds determine asteroid & comet distribution and transport, Neptune's manifolds may govern the Kuiper belt.



Transport Between Asteriod Belt and Kuiper Belt

- Intersections between L_1 and L_2 manifold structures between adjacent planets may provide a "highway" connecting the asteroid and Kuiper belts, where material can collect.



Conclusion: L_1 and L_2 Manifolds are Important!

- The invariant manifold structures associated to L_1 and L_2, as well as the homoclinic-heteroclinic ***dynamical channels*** connecting them, are ***fundamental tools*** that can aid in understanding mechanisms of transport throughout the solar system.