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Connecting orbits and invariant manifolds in the spatial three-body problem

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Introduction

- ***Goal***
  - Use dynamical systems techniques to identify key transport mechanisms and useful orbits for space missions.
- ***Outline***
  - Circular restricted three-body problem
  - Equilibrium points and invariant manifold structures
  - Construction of trajectories with prescribed itineraries
  - Connecting orbits, e.g., heteroclinic connections
  - Tours of Jupiter's moons

Introduction

Current research importance

- Development of some NASA mission trajectories, such as the recently launched *Genesis Discovery Mission*, and the upcoming *Jupiter Icy Moons Orbiter*
- Of current astrophysical interest for understanding the transport of solar system material (eg, how ejecta from Mars gets to Earth, etc.)

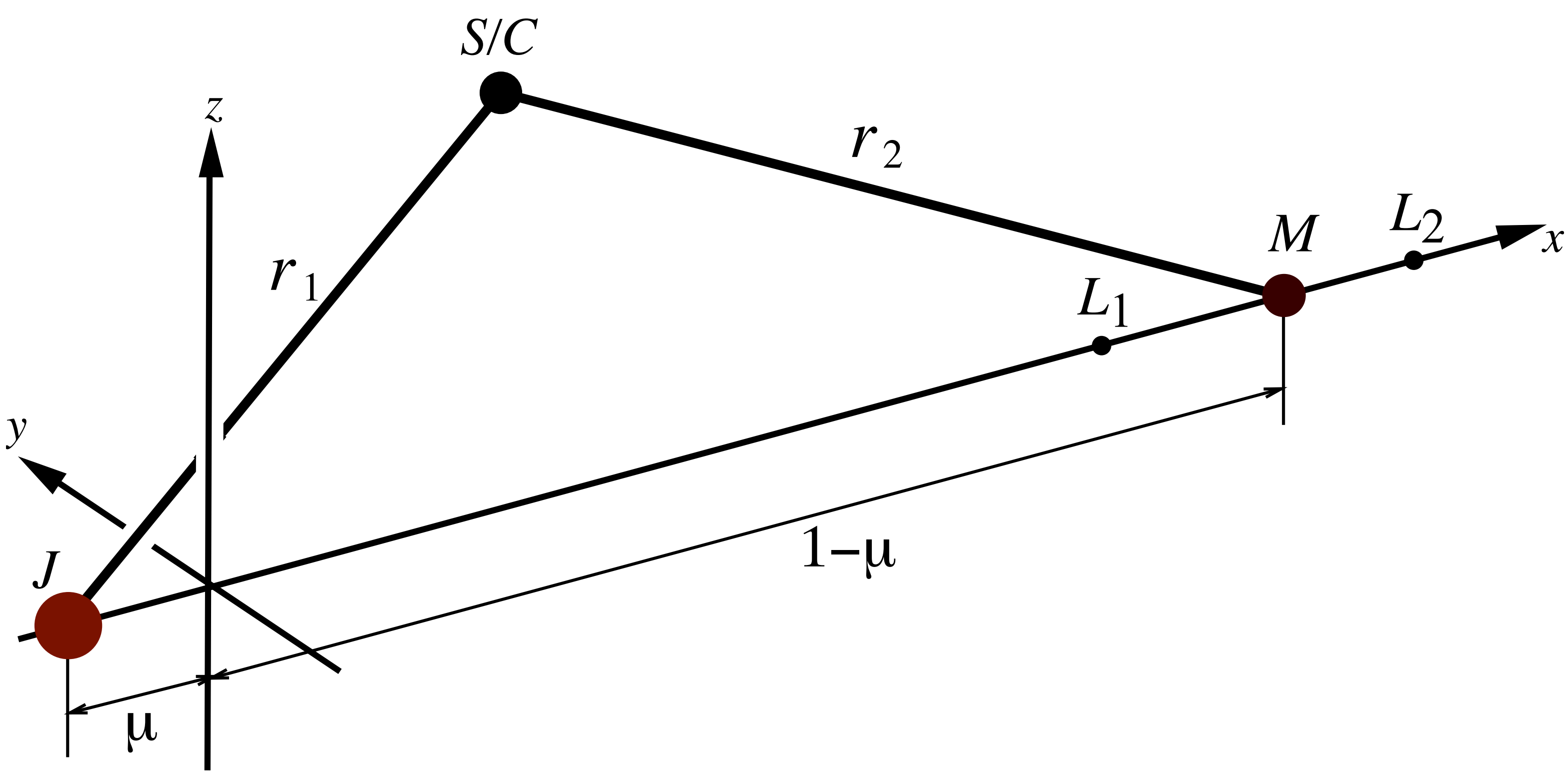
Three-Body Problem

- ***Circular restricted 3-body problem***
  - the two primary bodies move in circles; the much smaller third body moves in the gravitational field of the primaries, without affecting them
  - the two primaries could be Jupiter and a moon
  - the smaller body could be a spacecraft or asteroid
  - we consider the planar and spatial problems
  - there are five equilibrium points in the rotating frame, places of balance which generate interesting dynamics

Three-Body Problem

Circular restricted 3-body problem

- Consider the two unstable points on line joining the two main bodies – L_1, L_2



Equilibrium points – L_1, L_2

Three-Body Problem

- orbits exist around L_1 and L_2; both periodic and quasi-periodic
  - Lyapunov, halo and Lissajous orbits
- one can draw the invariant manifolds assoicated to L_1 (and L_2) and the orbits surrounding them
- these invariant manifolds play a key role in what follows

Three-Body Problem

- Equations of motion (planar):

$$\boxed{\ddot{x} - 2\dot{y} = -\bar{U}_x, \qquad \ddot{y} + 2\dot{x} = -\bar{U}_y}$$

  where

$$\bar{U} = -\frac{(x^2 + y^2)}{2} - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}.$$

- Have a first integral, the Hamiltonian energy, given by

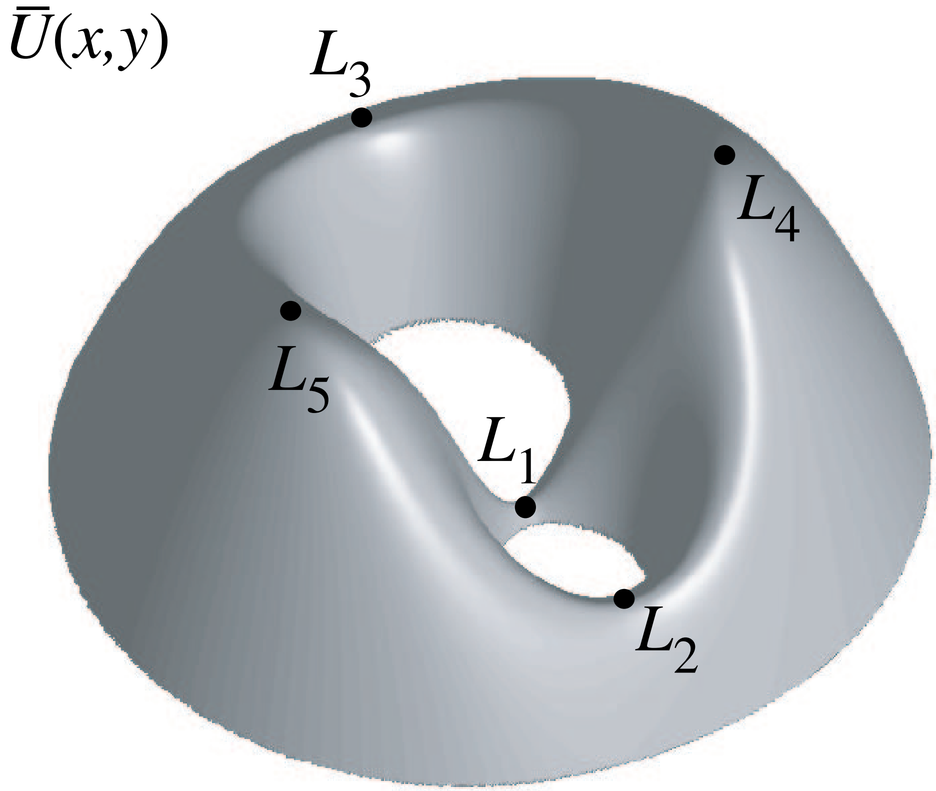
$$E(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \bar{U}(x, y).$$

- Energy manifolds are codimension 1 in the phase space.

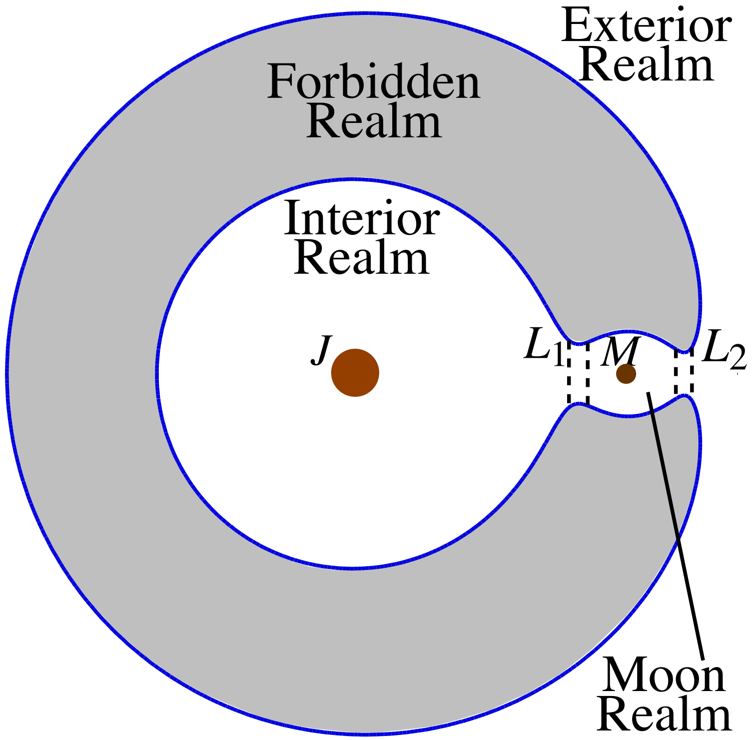
Realms of Possible Motion

- ***Effective potential***
  - In a rotating frame, the equations of motion describe a particle moving in an effective potential plus a magnetic field (goes back to work of Jacobi, Hill, etc).

Realms of Possible Motion



Effective potential



Level set shows accessible realms

Motion Near Equilibria

For saddles of rank 1

- Near equilibrium point, suppose linearized Hamiltonian vector field has eigenvalues $\pm i\omega_j$, $j = 1, \ldots, N-1$, and $\pm\lambda$.
- Assume the complexification is diagonalizable.
- Hamiltonian normal form theory tranforms Hamiltonian into a lowest order form:

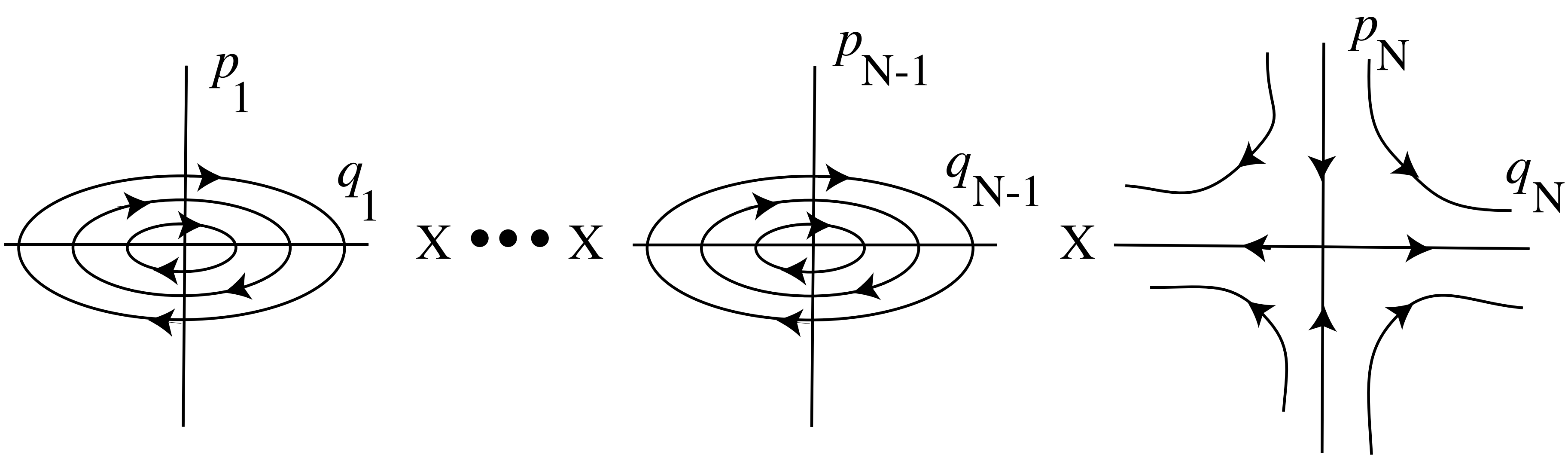
$$H(q,p) = \sum_{i=1}^{N-1} \frac{\omega_i}{2}\left(p_i^2 + q_i^2\right) + \lambda q_N p_N.$$

- Equilibrium point is of type center $\times \cdots \times$ center $\times$ saddle ($N-1$ centers).

Motion Near Equilibria

Multidimensional "saddle point"

- For fixed energy $H = h$, energy surface $\simeq S^{2N-2} \times \mathbb{R}$.
- Constants of motion:
  $I_j = q_j^2 + p_j^2, j = 1, \ldots, N-1$, and $I_N = q_N p_N$.



The ***N*** Canonical Planes

Motion Near Equilibria

- Normally hyperbolic invariant manifold at $q_N = p_N = 0$,

$$\mathcal{M}_h = \sum_{i=1}^{n-1} \frac{\omega_i}{2} \left(p_i^2 + q_i^2\right) = h > 0.$$

  Note that $\mathcal{M}_h \simeq S^{2N-3}$, not a single trajectory.

- Four “cylinders” of asymptotic orbits: the stable and unstable manifolds $W_\pm^s(\mathcal{M}_h), W_\pm^u(\mathcal{M}_h)$, which have the structure $S^{2N-3} \times \mathbb{R}$.

Flow Near Equilibria

- Dynamics near L_1 & L_2 in spatial problem: **saddle** $\times$ **center** $\times$ **center**.
- Hamiltonian for linearized equations has eigenvalues $\pm\lambda, \pm i\nu$, and $\pm i\omega$, where $\nu \neq \omega$,
- Change of coordinates yields

$$H_2 = \lambda q_1 p_1 + \frac{\nu}{2}(q_2^2 + p_2^2) + \frac{\omega}{2}(q_3^2 + p_3^2).$$

- For fixed energy $H = h$, energy surface $\simeq S^4 \times \mathbb{R}$.
- Constants of motion: $q_1 p_1$, $q_2^2 + p_2^2$ and $q_3^2 + p_3^2$.

Flow Near Equilibria

- Normally hyperbolic invariant manifold at $q_1 = p_1 = 0$,

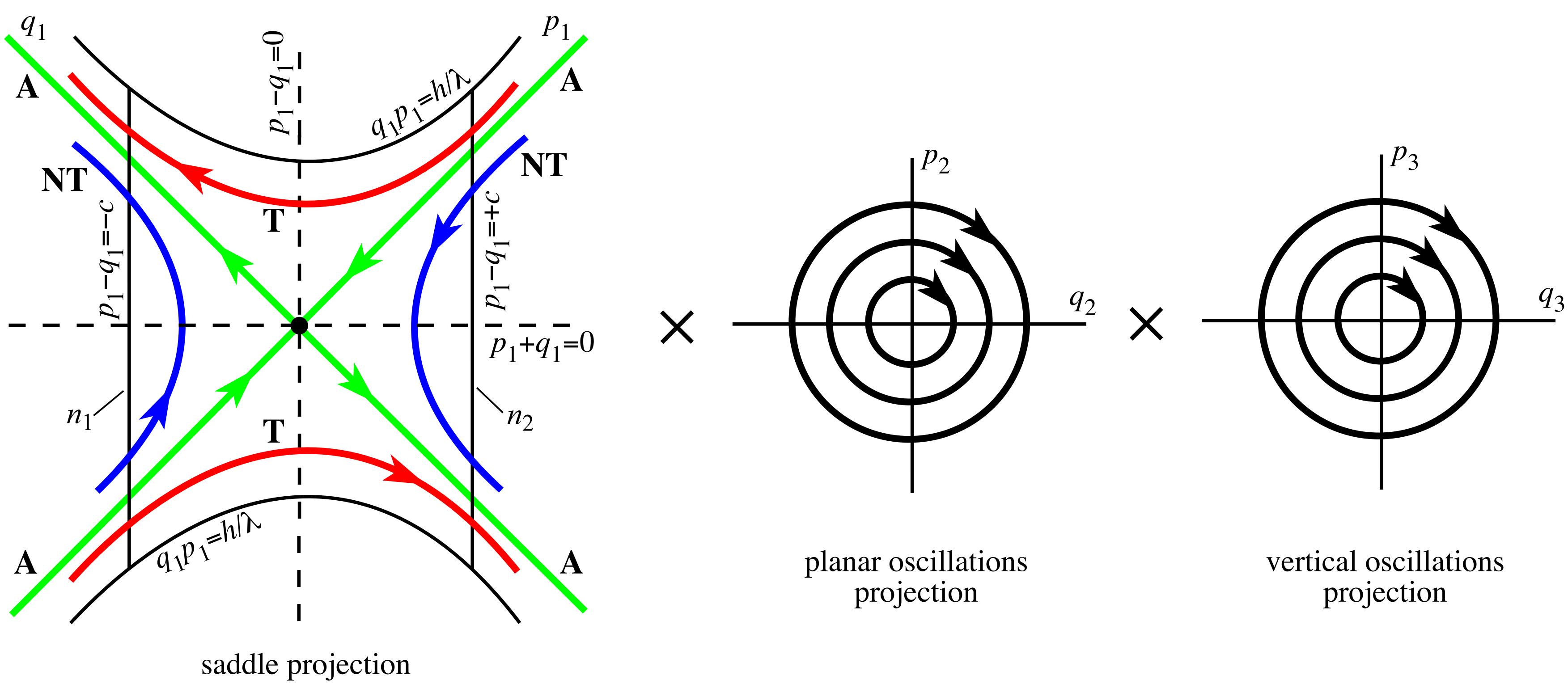
$$\mathcal{M}_h = \frac{\nu}{2}(q_2^2 + p_2^2) + \frac{\omega}{2}(q_3^2 + p_3^2) = h > 0.$$

  Note that $\mathcal{M}_h \simeq S^3$, not a single trajectory.

- Four "cylinders" of asymptotic orbits: the stable and unstable manifolds $W^s_\pm(\mathcal{M}_h), W^u_\pm(\mathcal{M}_h)$, which have the structure $S^3 \times \mathbb{R}$.

Flow Near Equilibria

- **B** : **bounded orbits** (periodic/quasi-periodic): S^3 (3-sphere)
- **A** : **asymptotic orbits** to 3-sphere: $S^3 \times I$ (**"tubes"**)
- **T** : **transit** and **NT** : **non-transit** orbits.



The flow in the equilibrium region.

Flow Near Equilibria

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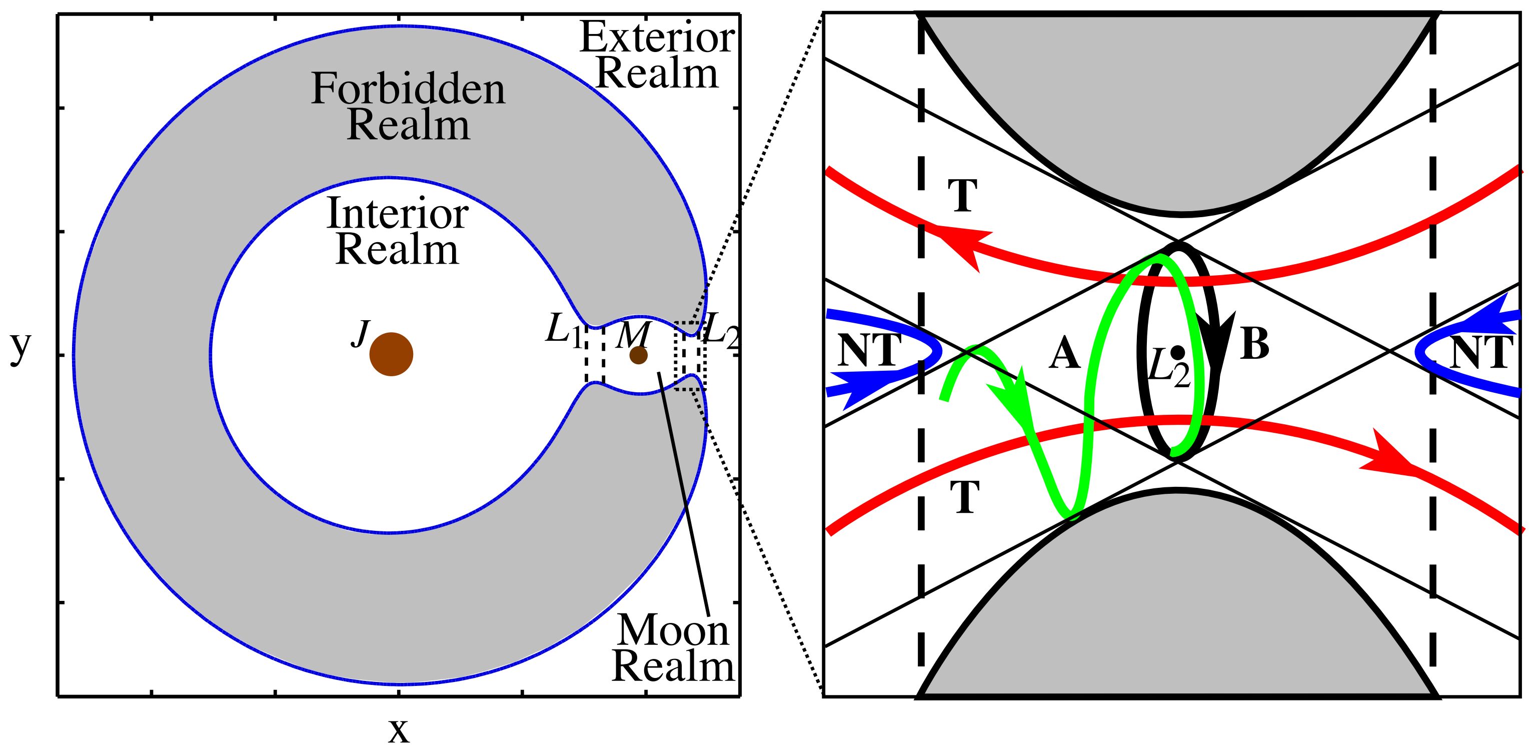
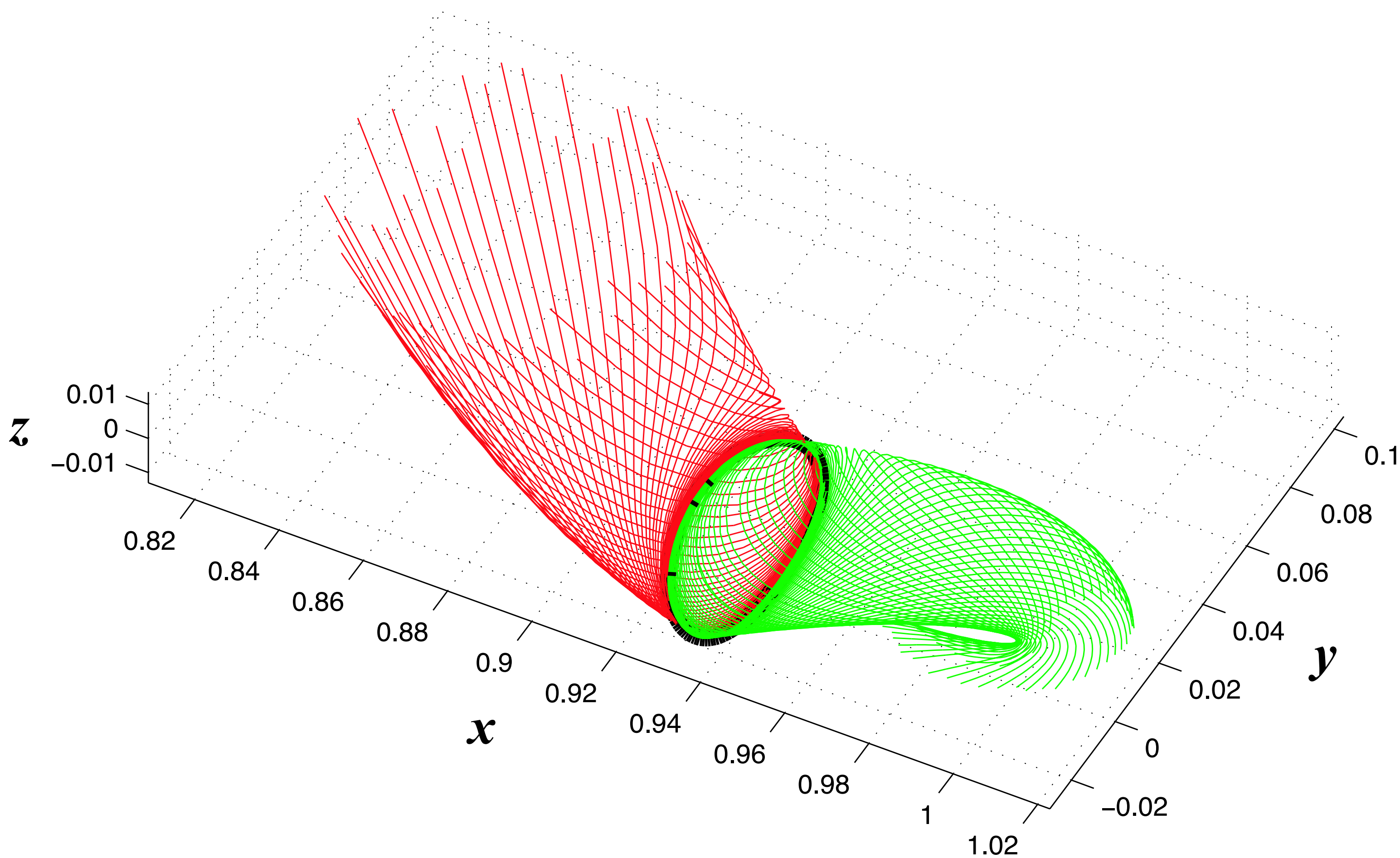


Projection to configuration space.

Transport Between Realms

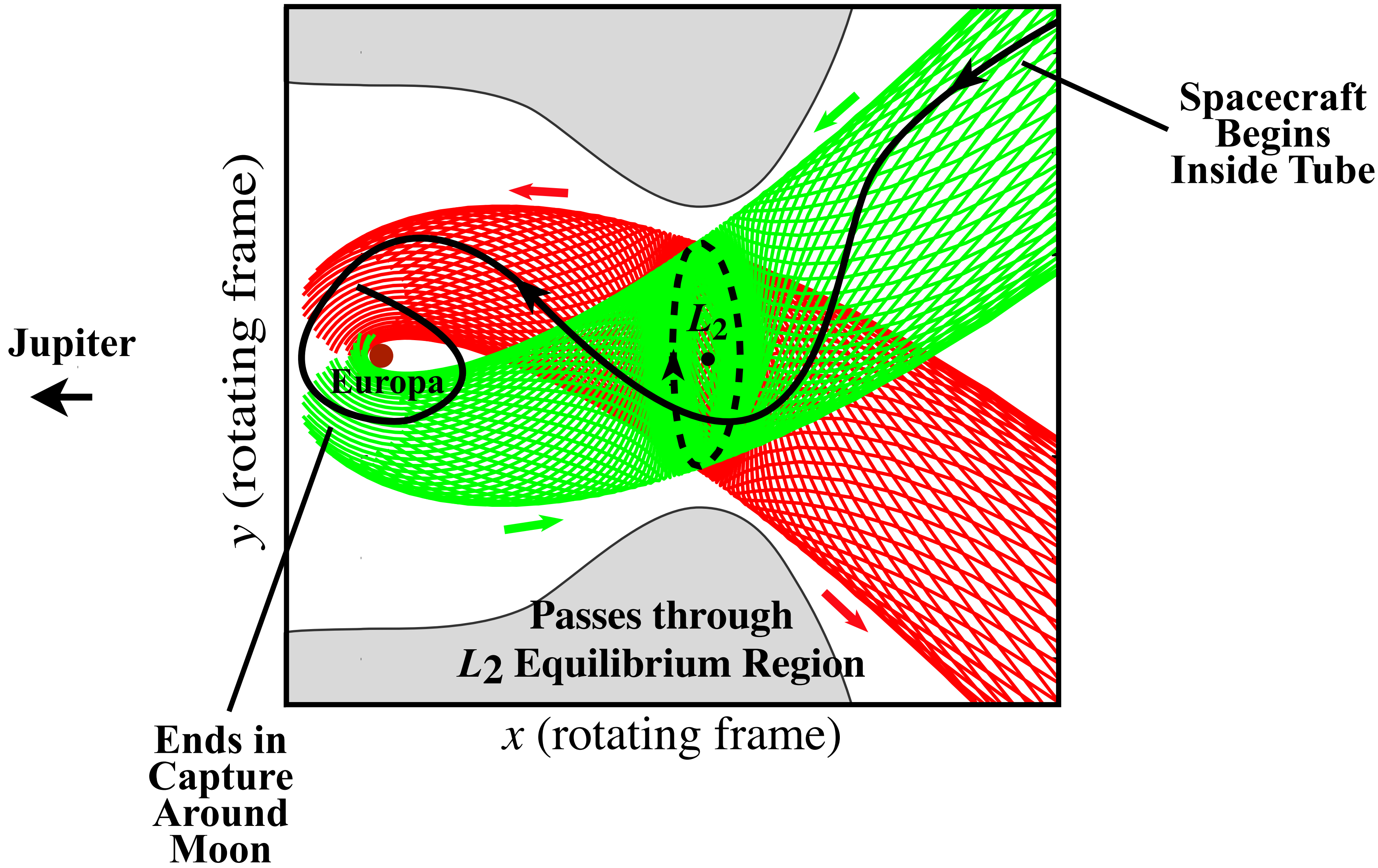
- Asymptotic orbits form **4D invariant manifold tubes** $(S^3 \times I)$ in 5D energy surface.
- red = unstable, green = stable



Transport Between Realms

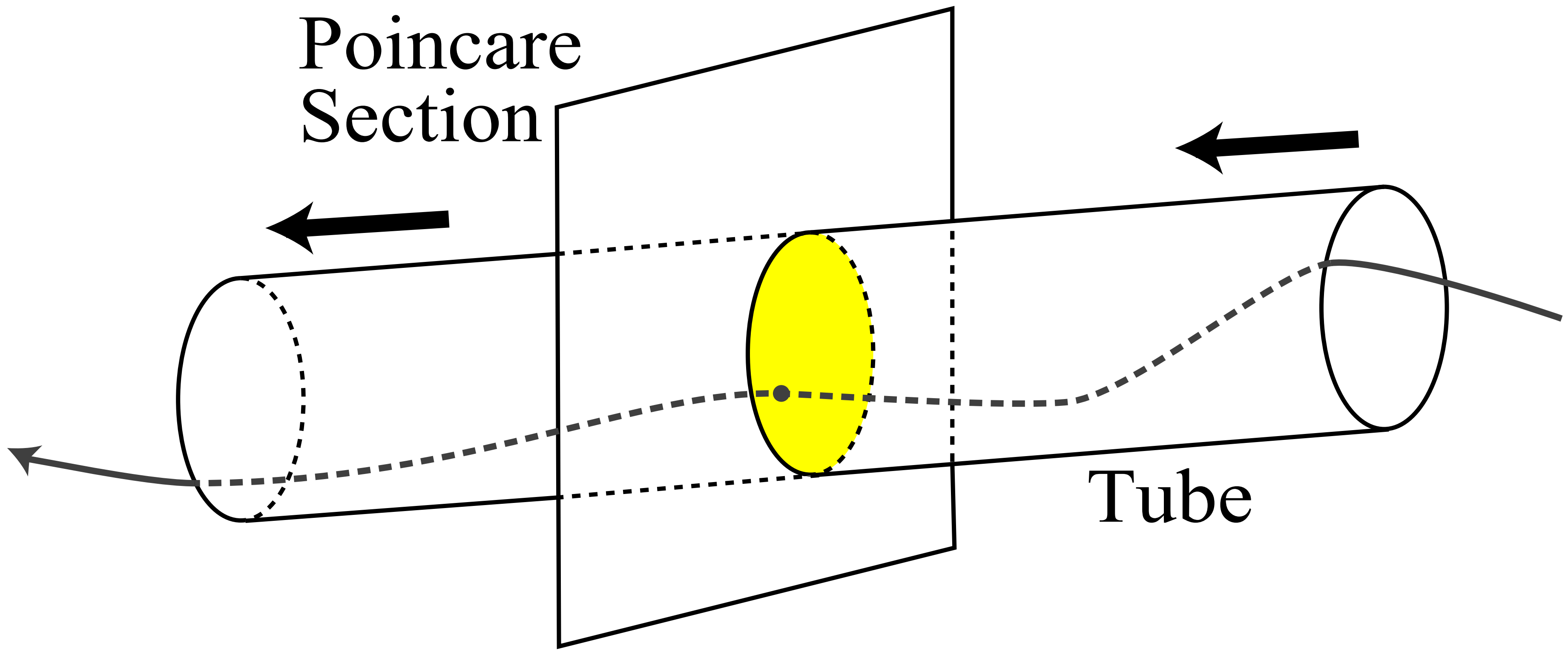
- These manifold tubes play an important role in governing what orbits approach or depart from a moon (**transit orbits**)
- and orbits which do not (**non-transit orbits**)
- transit possible for objects "inside" the tube, otherwise no transit — this is important for transport issues

Transport Between Realms



Transport Between Realms

- Transit orbits can be found using a **Poincaré section** transversal to a tube.

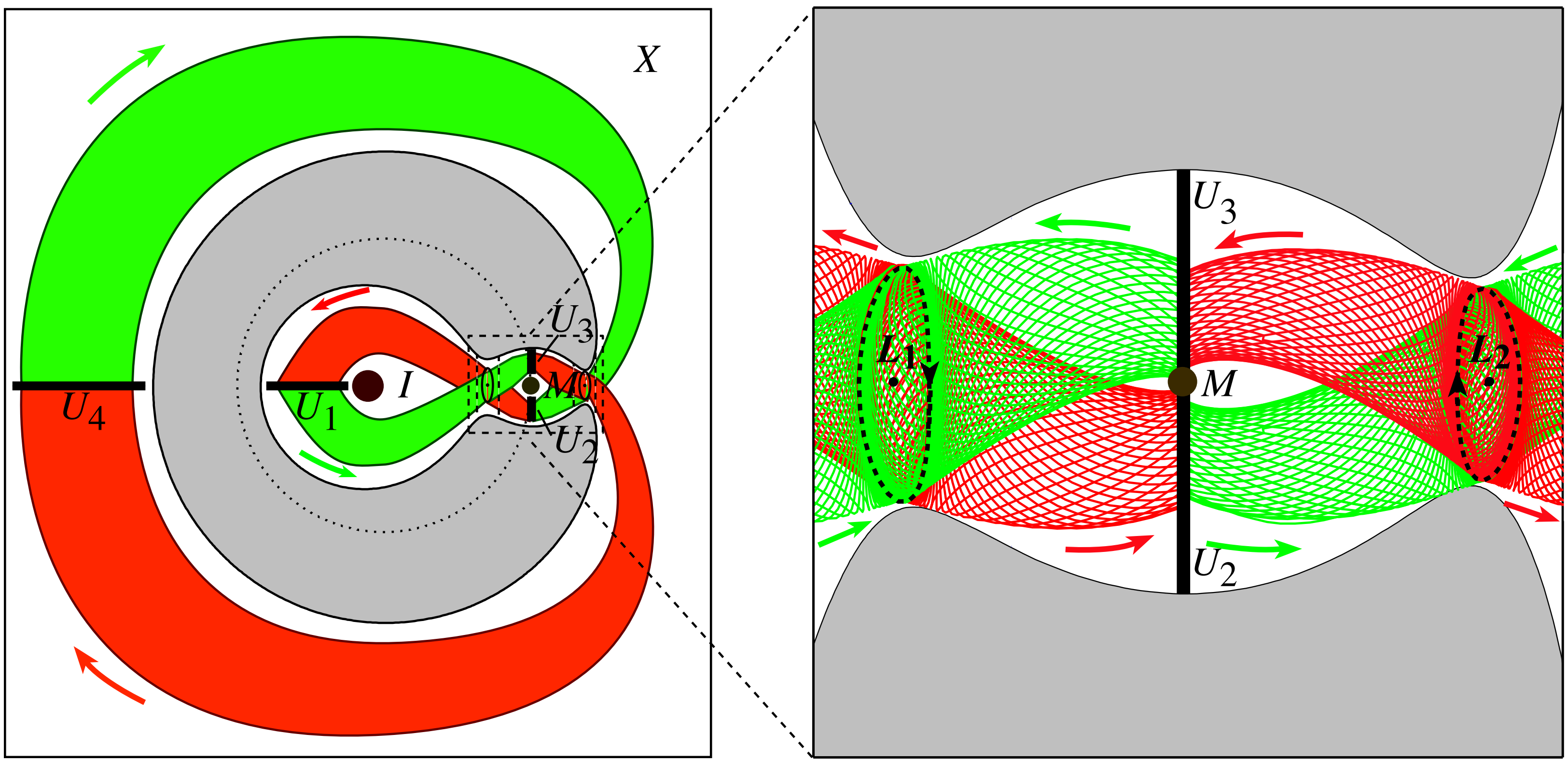


Construction of Trajectories

- One can systematically construct new trajectories, which use little fuel.
  - by linking stable and unstable manifold tubes in the right order
  - and using Poincaré sections to find trajectories "inside" the tubes
- One can construct trajectories involving multiple 3-body systems.

Construction of Trajectories

- For a single 3-body system, we wish to **link** invariant manifold tubes to construct an orbit with a **desired itinerary**
- Construction of $(X; M, I)$ orbit.



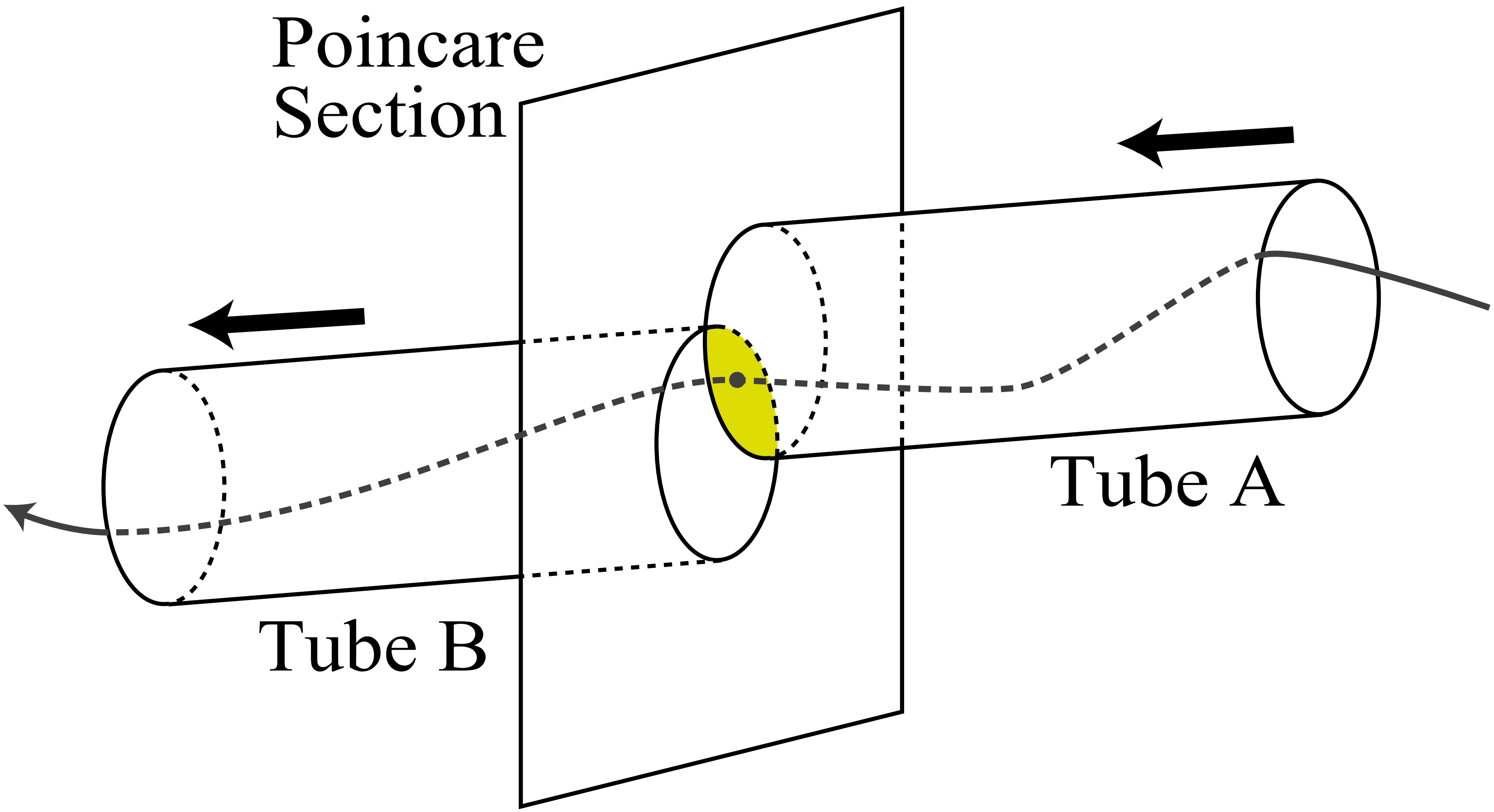
The tubes connecting the X,M, and I regions.

Construction of Trajectories

- First, integrate two tubes until they pierce a common **Poincaré section** transversal to both tubes.
- Second, pick a point in the region of intersection and integrate it forward and backward.

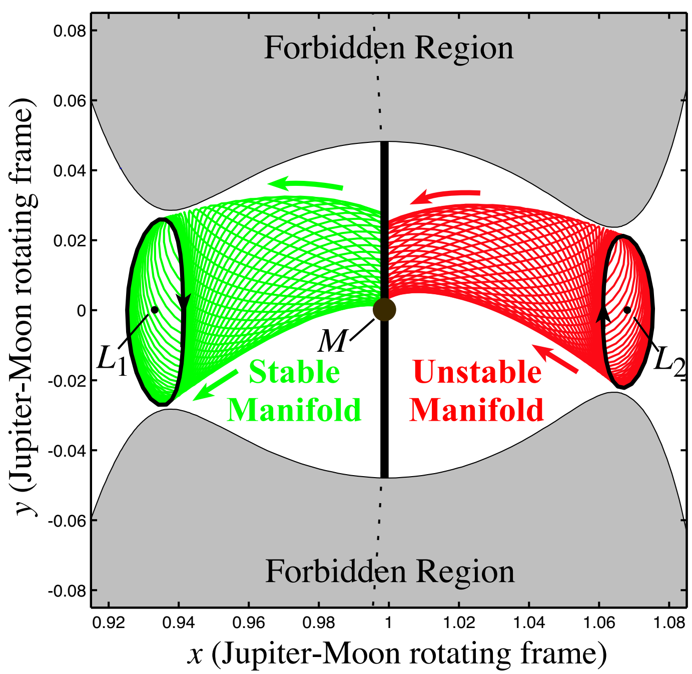
Construction of Trajectories

- Integrate two tubes
- Integrate a point in the region of intersection

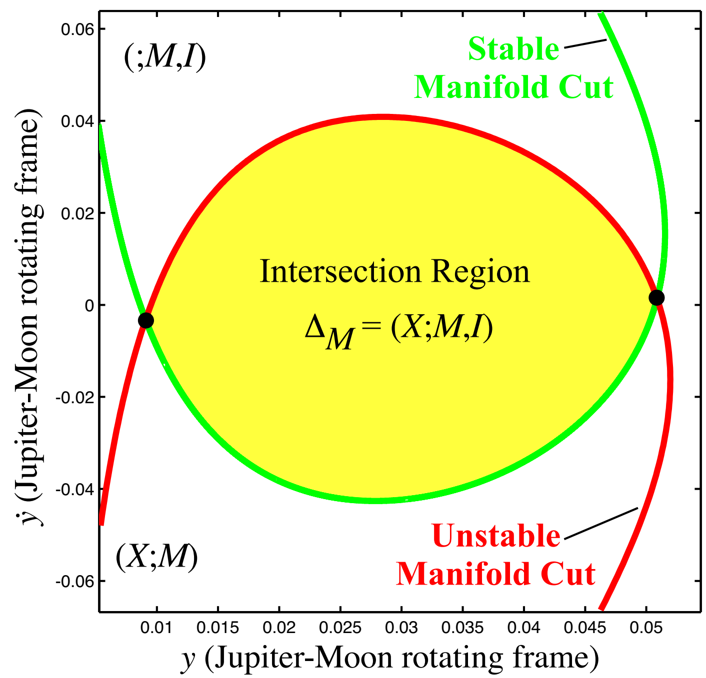


Construction of Trajectories

- **Planar**: tubes $(S \times I)$ separate transit/non-transit orbits.
- **Red curve (S^1)** : slice of L_2 **unstable manifold**
  Green curve (S^1) : slice of L_1 **stable manifold**
- Any point inside the intersection region Δ_M is a $(X; M, I)$ orbit.



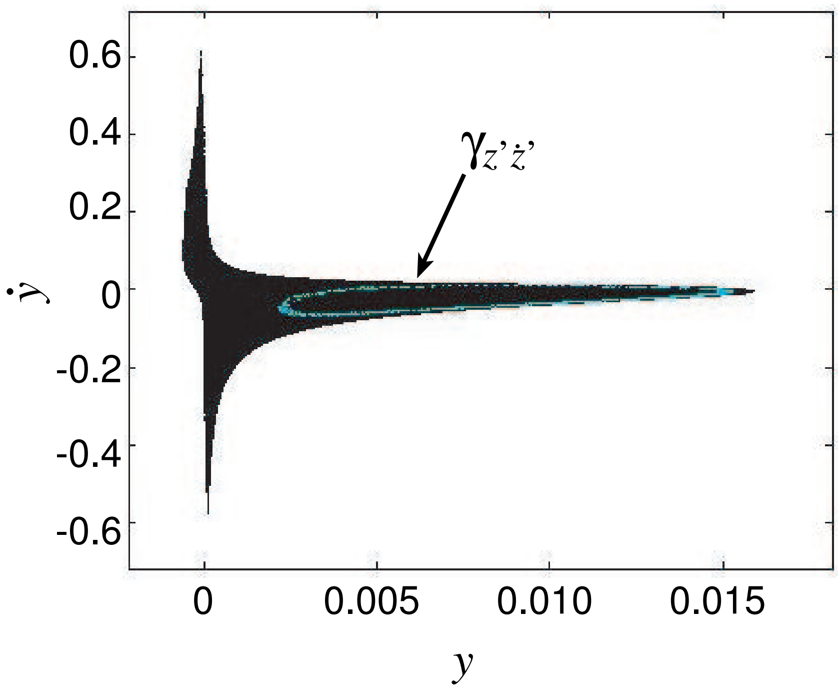
Tubes intersect in position



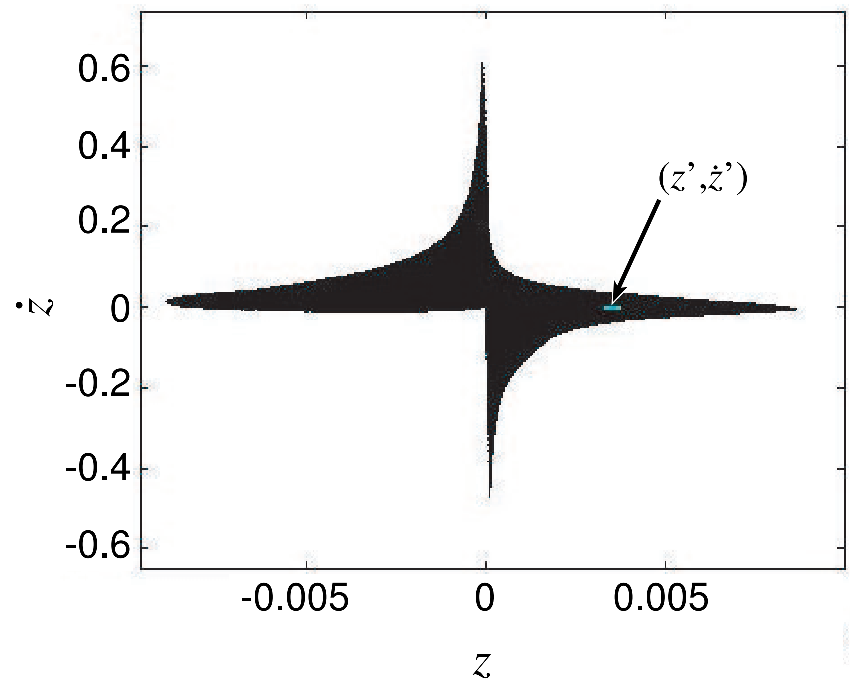
Poincaré section of intersection

Construction of Trajectories

- **Spatial**: Invariant manifold tubes $(S^3 \times I)$
- Poincaré **slice** is a topological **3-sphere** S^3 in $\mathbb{R}^4$.
  - S^3 looks like **disk** $\times$ **disk**: $\xi^2 + \dot{\xi}^2 + \eta^2 + \dot{\eta}^2 = r^2 = r_\xi^2 + r_\eta^2$
- Find $(X; M)$ orbit.



$(y, \dot{y})$ Plane

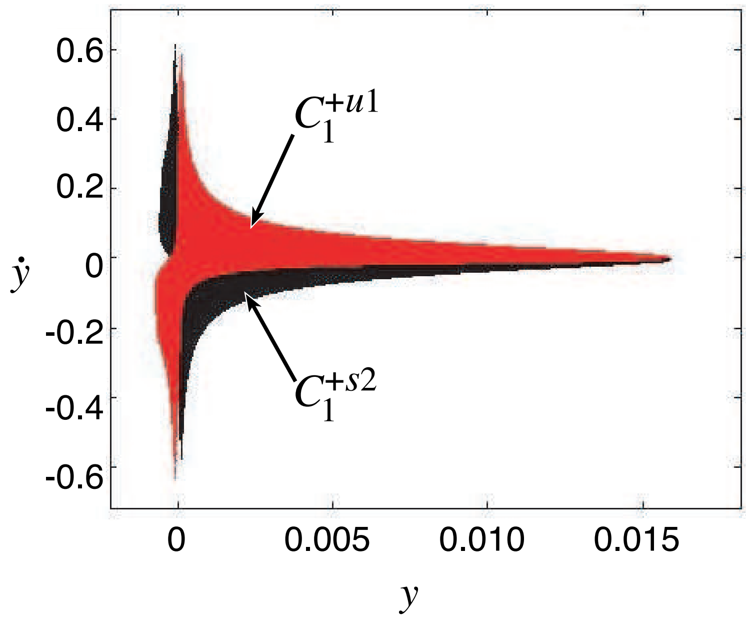


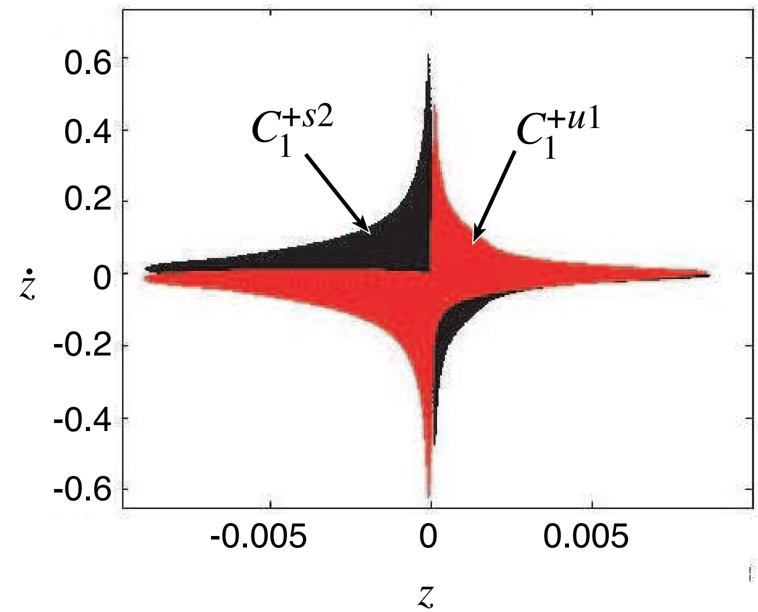
$(z, \dot{z})$ Plane

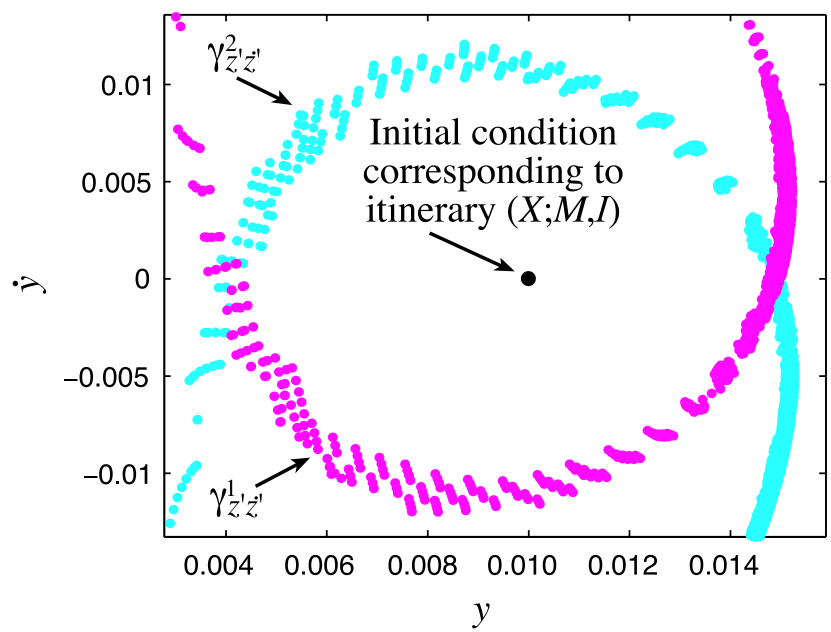
Construction of Trajectories

- Similarly, while the cut of the **stable** manifold tube is S^3, its projection on $(y, \dot{y})$ plane is a **curve** for $z = c, \dot{z} = 0$.
- Any point inside this **curve** is a (M, I) orbit.
- Hence, any point inside the **intersection region** Δ_M is a $(X; M, I)$ orbit.

Construction of Trajectories

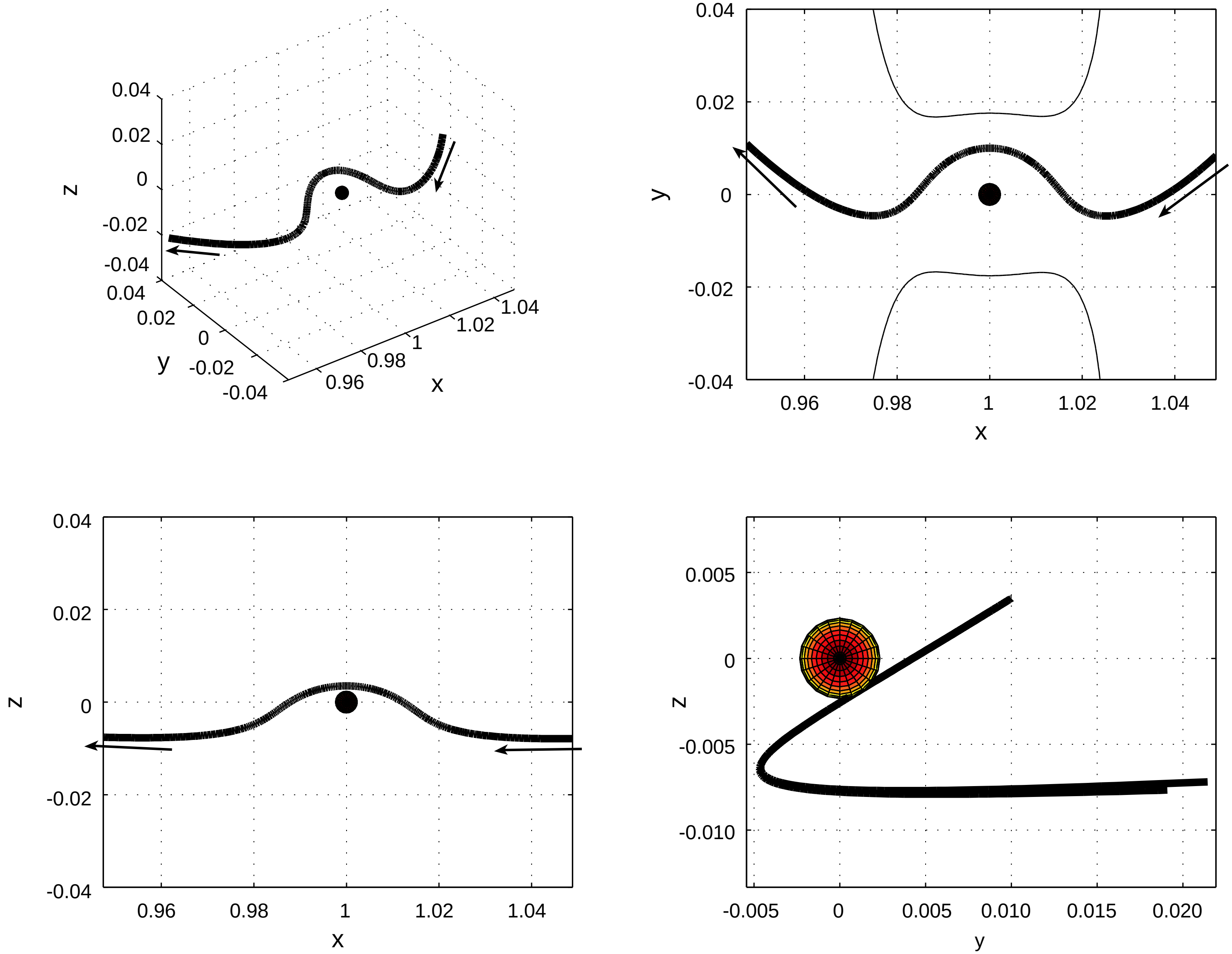






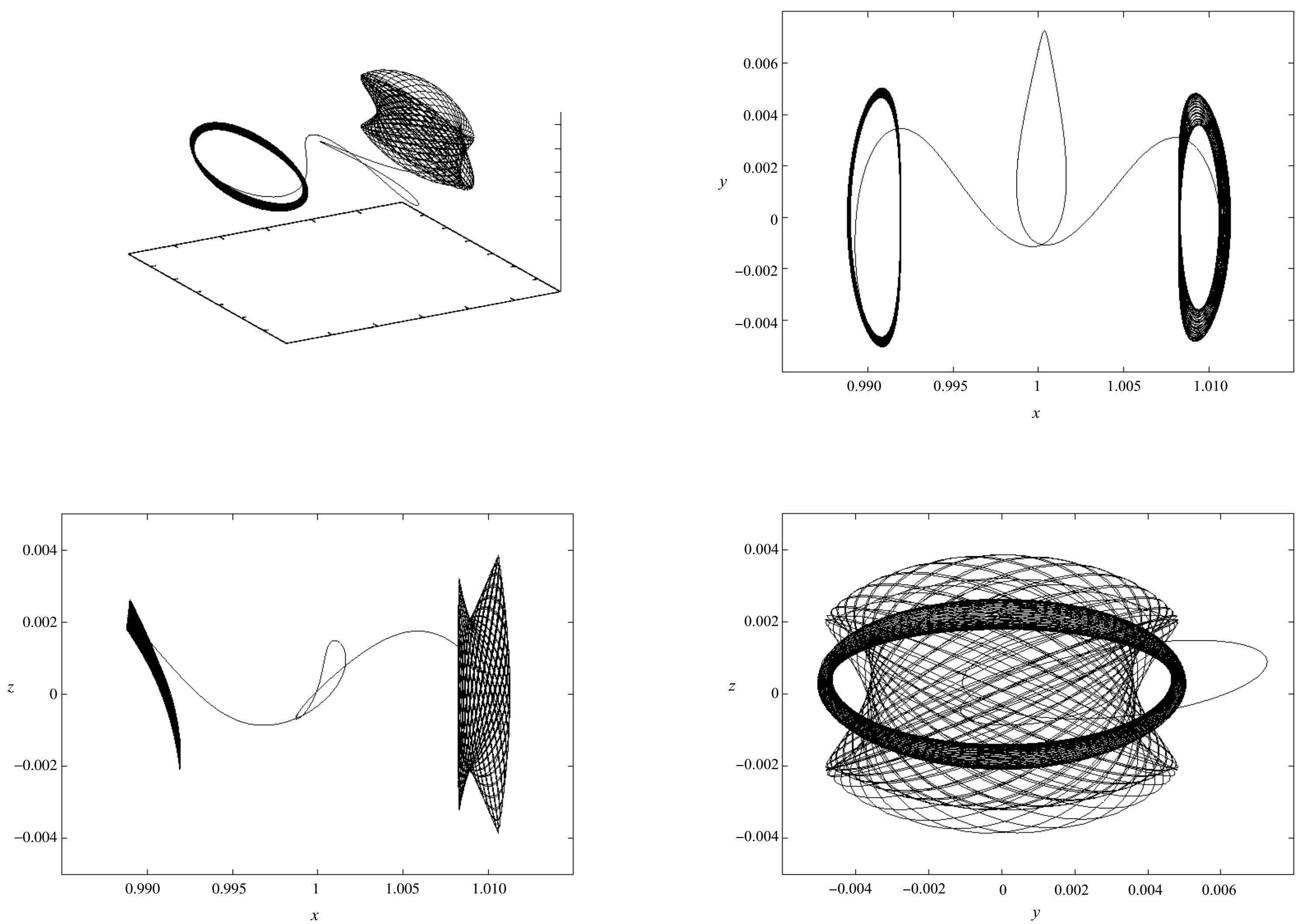
Intersection Region

Construction of Trajectories



Construction of an (X, M, I) orbit

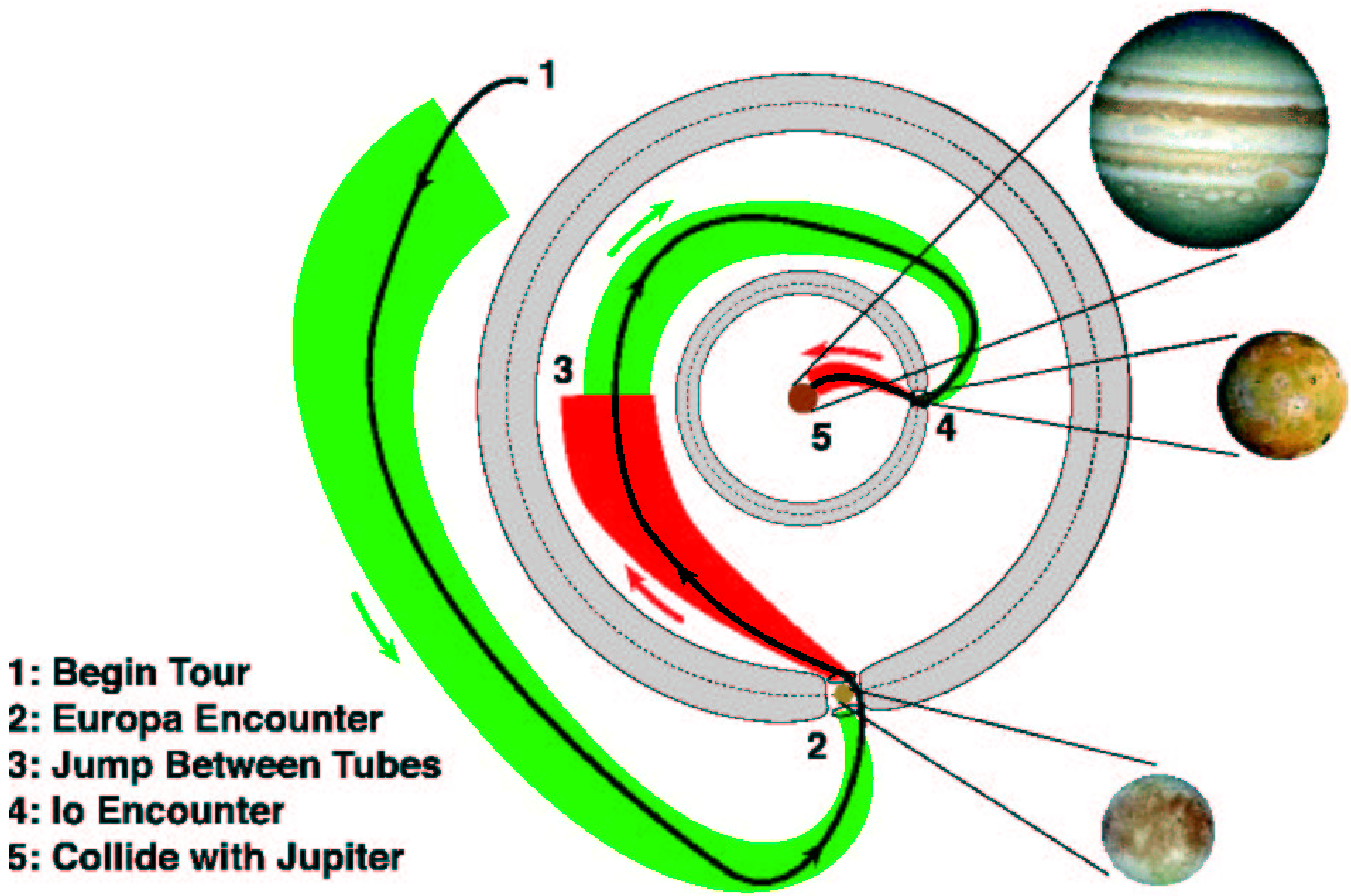
Connecting Orbits



An L_1-L_2 heteroclinic connection

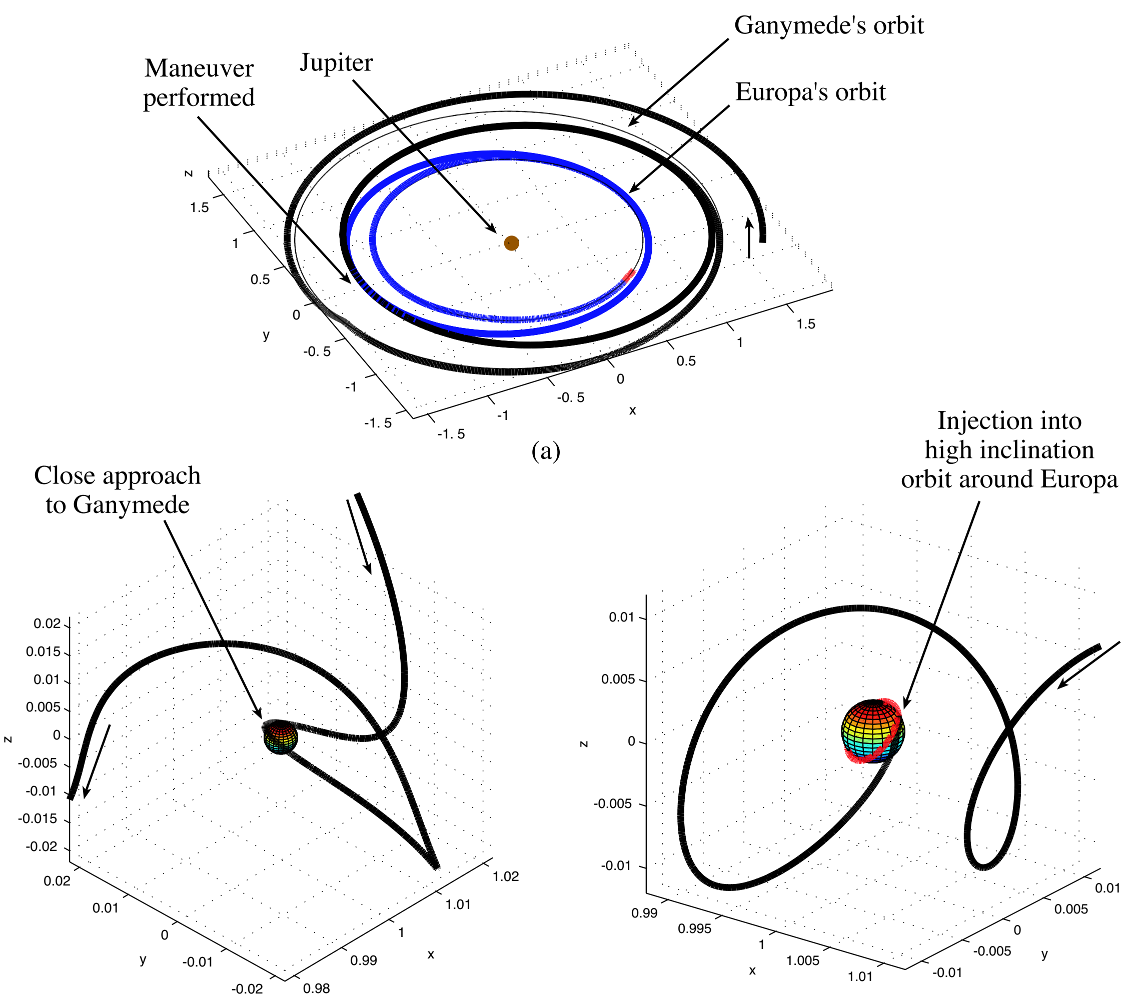
Tours of Jupiter's Moons

- ***Tours of planetary satellite systems.***
  - *Example 1:* Europa → Io → Jupiter



Tours of Jupiter's Moons

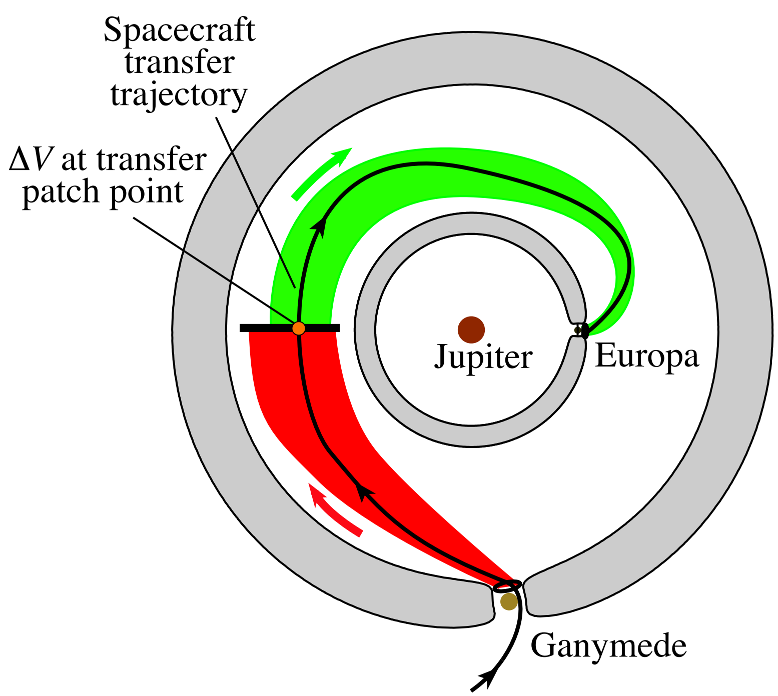
- *Example 2:* Ganymede → Europa → injection into Europa orbit



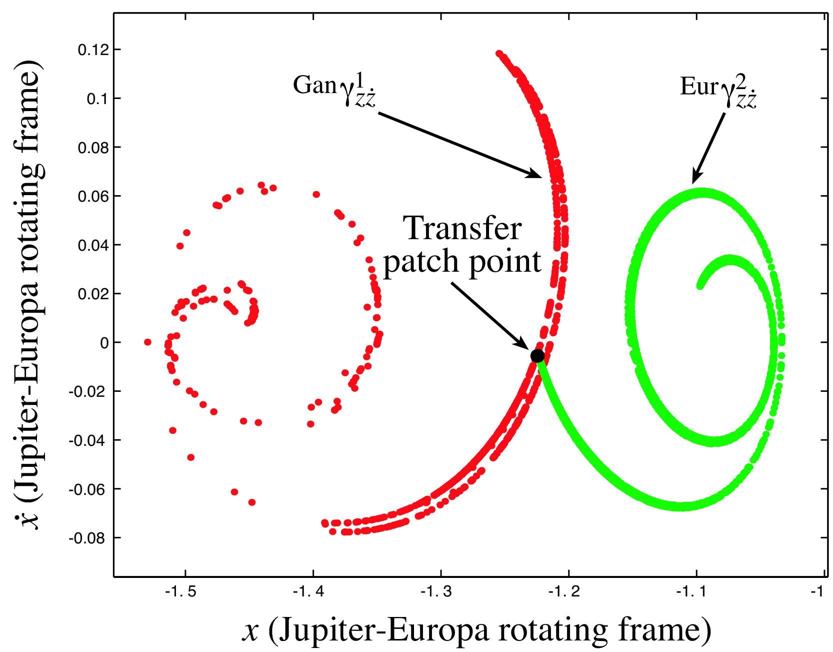
(a) (b) (c)

Tours of Jupiter's Moons

- The **Petit Grand Tour** can be constructed as follows:
  - Approximate 4-body system as 2 nested **3-body systems**.
  - Choose an appropriate Poinaré section.
  - Link the invariant manifold tubes in the proper order.
  - Integrate initial condition (patch point) in the 4-body model.



Look for intersection of tubes



Poincaré section at intersection

Some References

- Gómez, G., W.S. Koon, M.W. Lo, J.E. Marsden, J. Masdemont and S.D. Ross [2001] ***Connecting orbits and invariant manifolds in the spatial three-body problem.*** submitted to *Nonlinearity*.
- Gómez, G., W.S. Koon, M.W. Lo, J.E. Marsden, J. Masdemont and S.D. Ross [2001] ***Invariant manifolds, the spatial three-body problem and space mission design.*** *AAS/AIAA Astrodynamics Specialist Conference*.
- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000] ***Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics.*** *Chaos* 10(2), 427–469.

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