Lagrangian coherent structures in fluid dynamics

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SES, October 2011

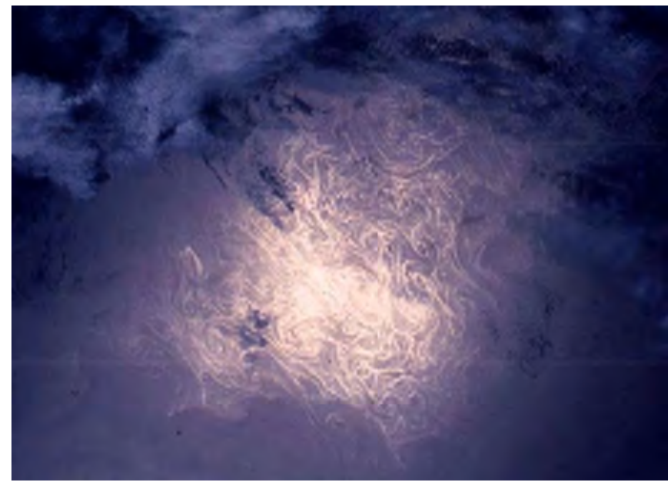
MultiSTEPS: MultiScale Transport in Environmental & Physiological Systems, www.multisteps.esm.vt.edu

Lagrangian coherent structures – coherent structures moving with the fluid

Some vortex examples from geophysical fluids

hurricanes

tornados



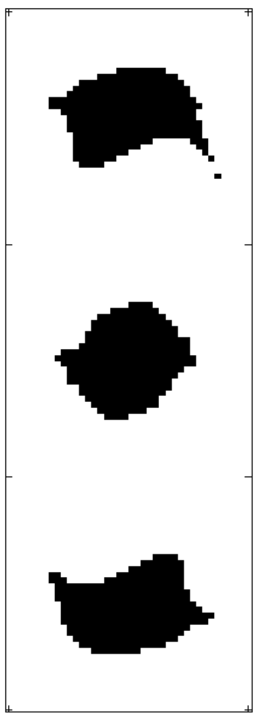
eddies

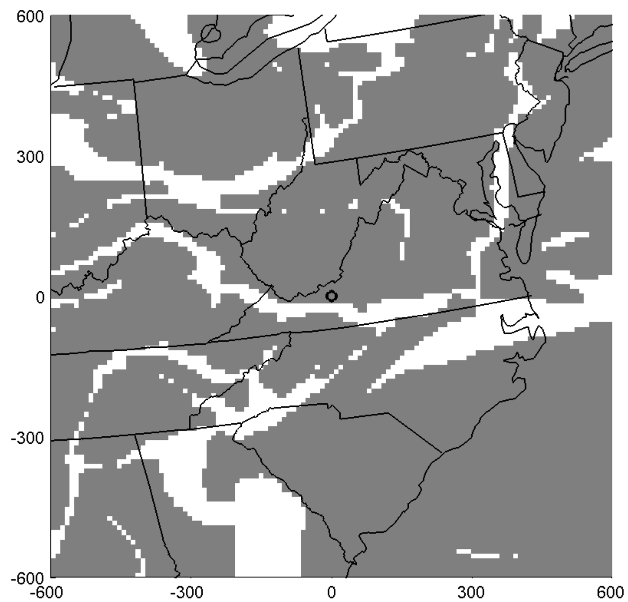
Why Lagrangian coherent structures?

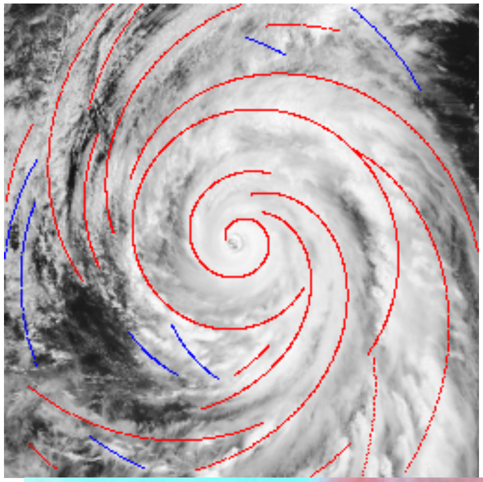
- Natural flow visualization
- Insight for design and control
- Application to fluid and non-fluid systems

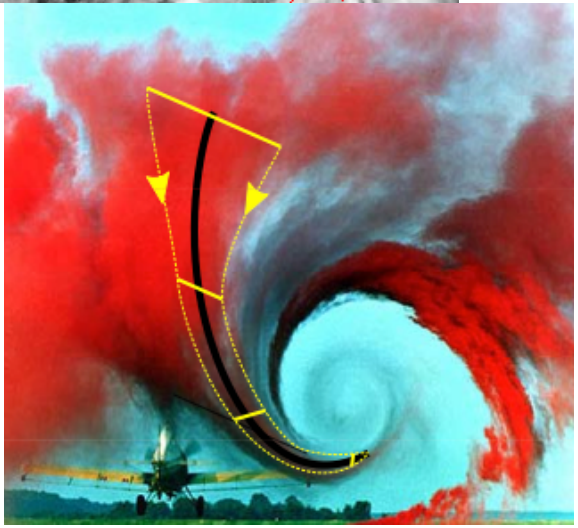
Lagrangian coherent structures

Two strategies from *dynamical systems*:
(1) Identify their `*skeleton*' or *boundaries*
(2) identify the structures *directly*









Lagrangian coherent structures

Two strategies from *dynamical systems*:
(1) Identify their `*skeleton*' or *boundaries*
(2) identify the structures *directly*

For (1):

For time-periodic or steady fluid velocity fields, identify periodic orbits + their stable & unstable manifolds.

But for *realistic* flows,
Time-independent and aperiodic, data-driven, finite-time,
so need something else?

For (2):

Discretize the *flow map* over timescale of interest;
determine coherent regions via eigenmodes, graph methods

Atmosphere: Antarctic polar vortex

Atmosphere: Antarctic polar vortex

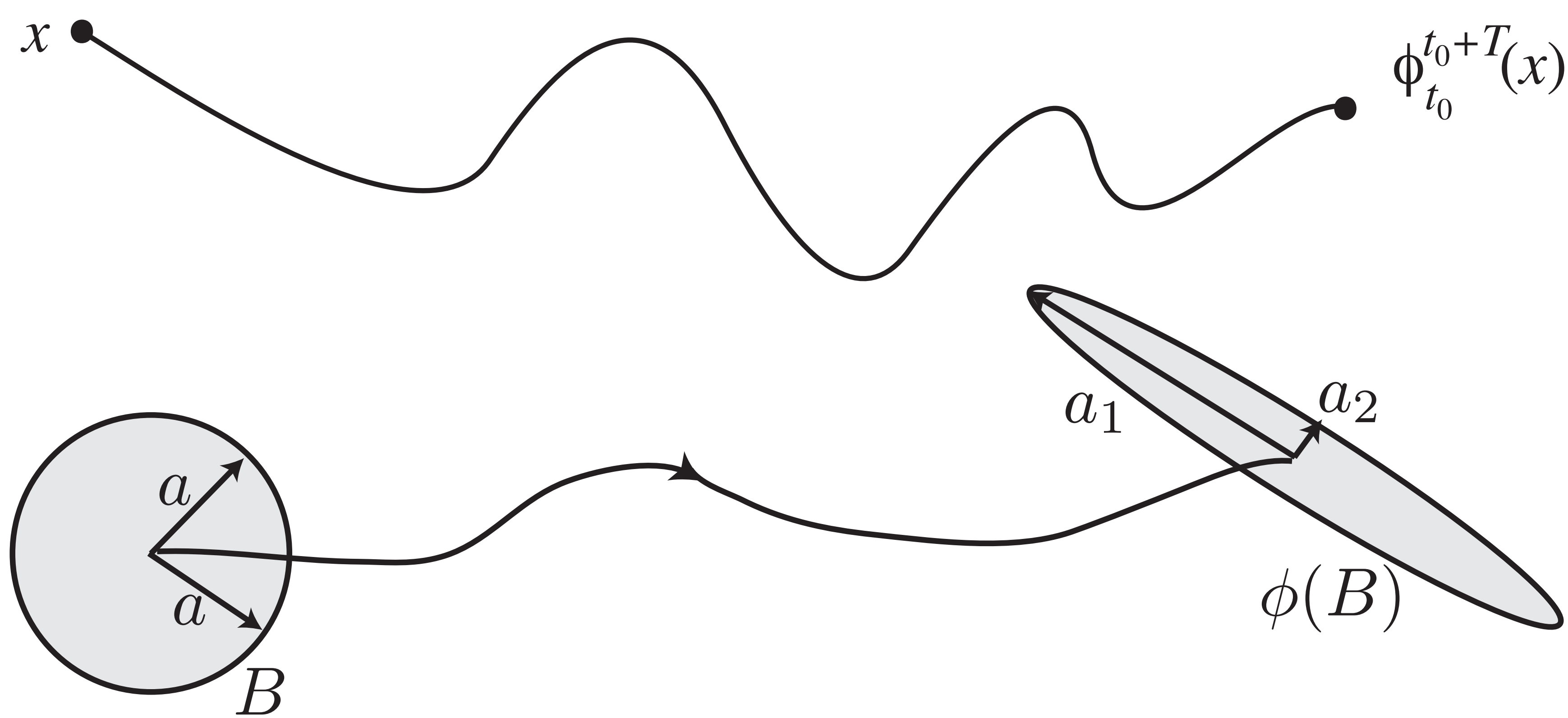
Atmosphere: continental U.S.

Periodic velocity field

- If $\mathcal{M}$= fluid domain, the flow map,

$$\phi_t^{t+T} : \mathcal{M} \longrightarrow \mathcal{M},$$

takes points $x \mapsto \phi_t^{t+T}(x)$ to their location after time T



Periodic velocity field

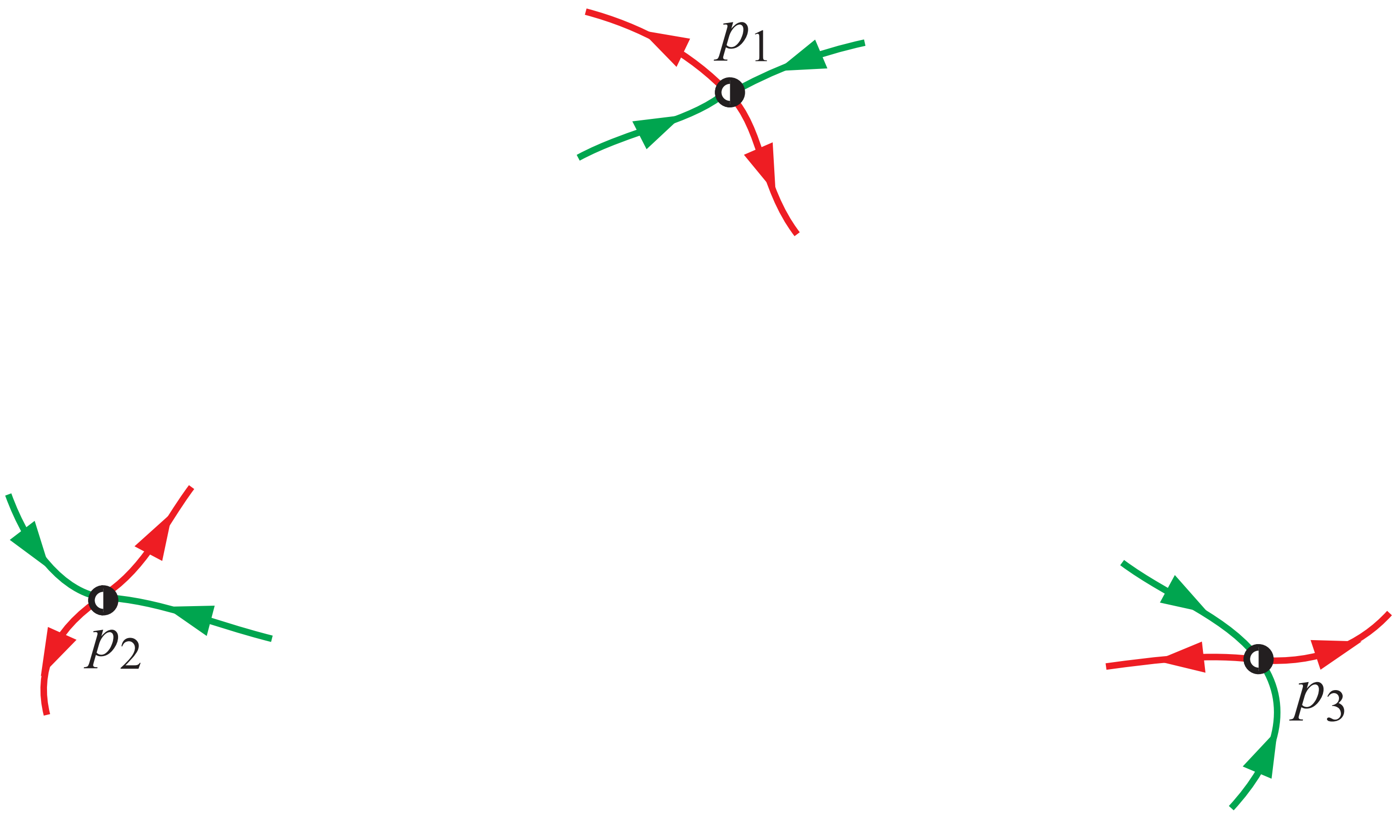
- Suppose our velocity field is periodic with period T and we consider the flow map $f = \phi_t^{t+T}$ over one period

- To understand the transport of points under the map f, consider **invariant manifolds of saddle fixed points**

- Let $p_i, i = 1, ..., N_p$, denote a collection of saddle-type hyperbolic fixed points for f.

Partition phase space into coherent regions

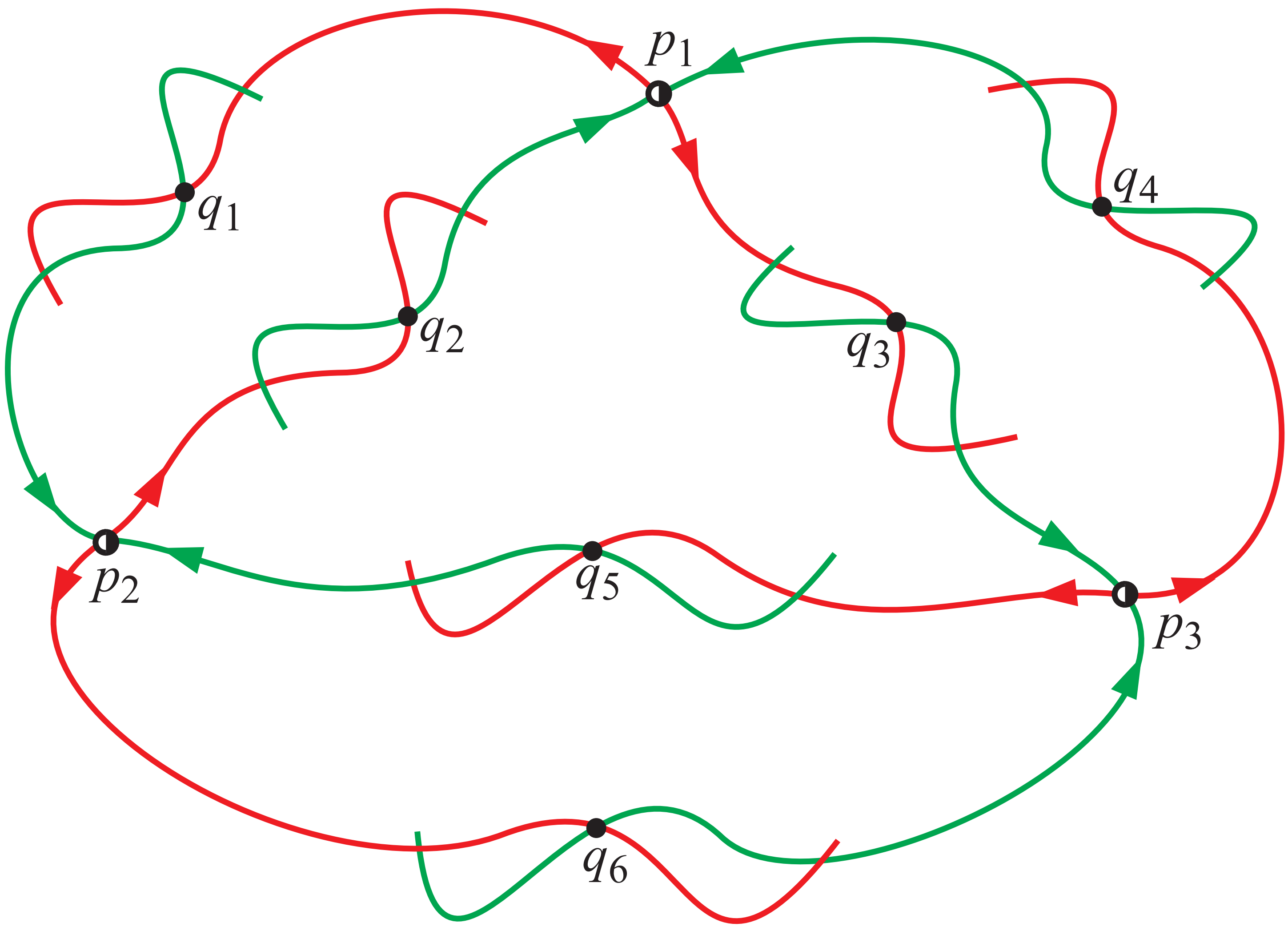
- Natural way to partition phase space
  - Pieces of $W^u(p_i)$ and $W^s(p_i)$ partition $\mathcal{M}$.



Unstable and stable manifolds in **red** and **green**, resp.

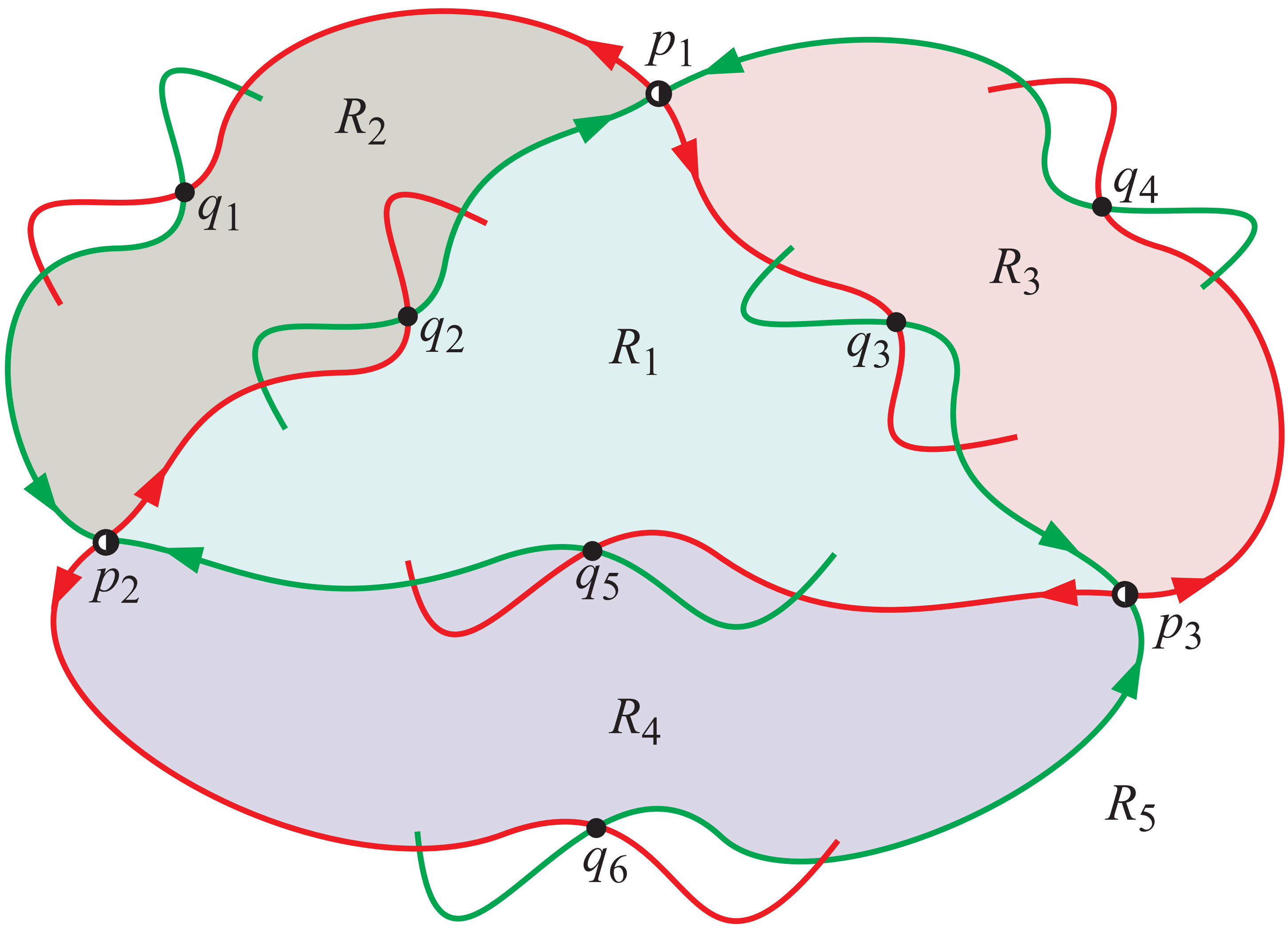
Partition phase space into coherent regions

- Intersection of unstable and stable manifolds define **boundaries**.



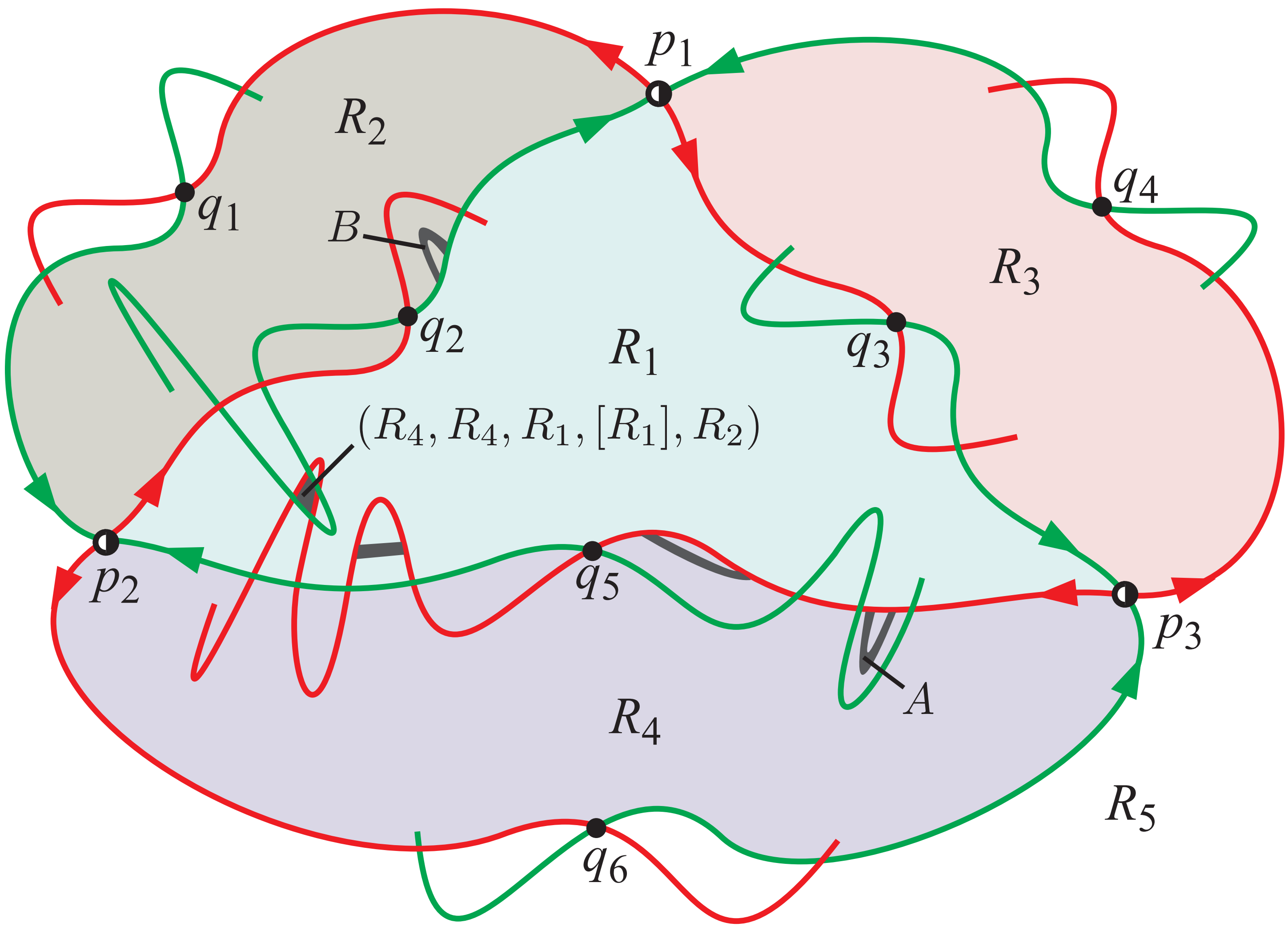
Partition phase space into coherent regions

- These boundaries divide the phase space into **coherent regions**.



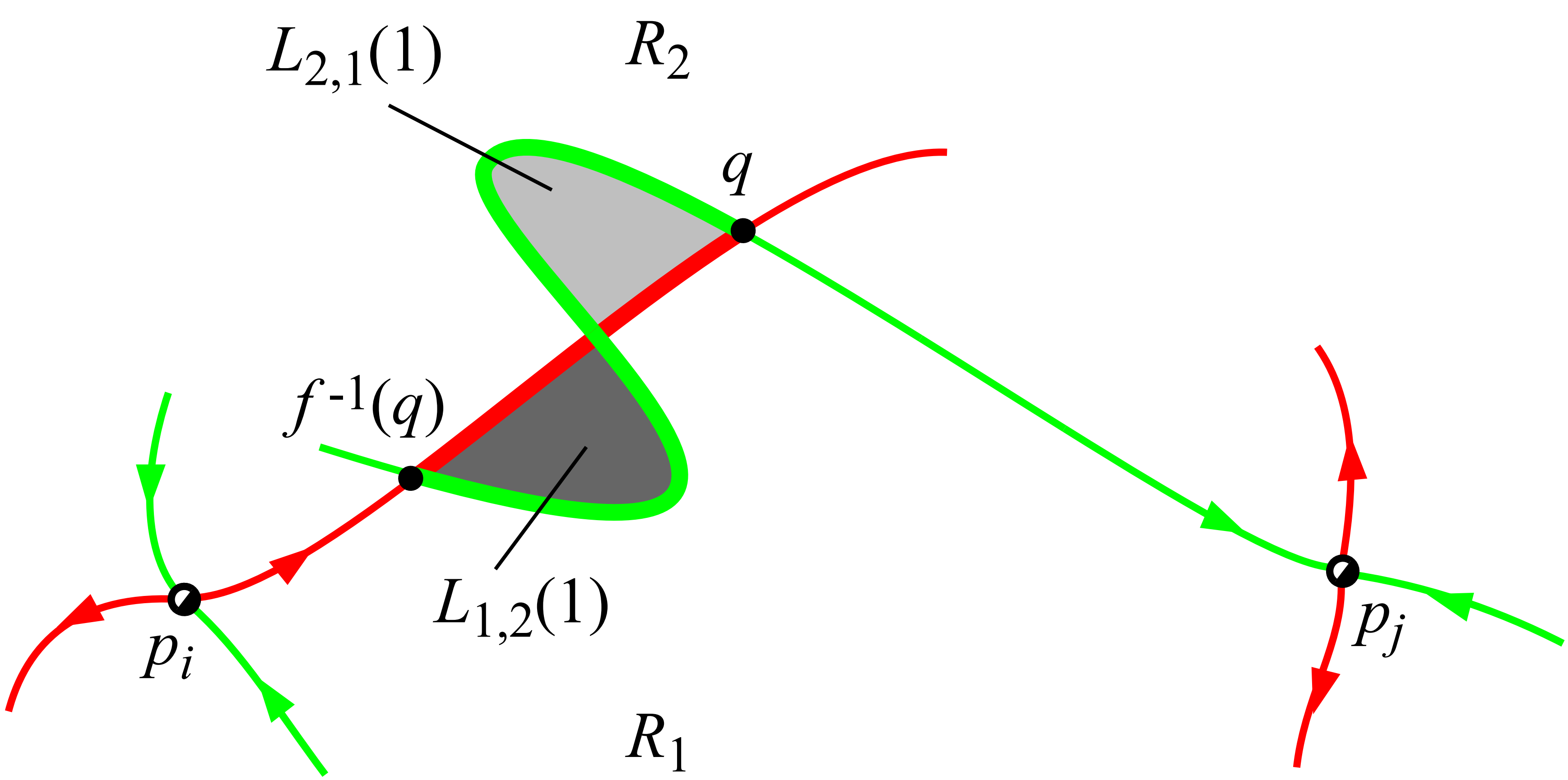
Label mobile subregions: ‘atoms’ of transport

- Can label mobile subregions based on their past and future whereabouts under one iterate of the map, e.g., $(\ldots, R_4, R_4, R_1, [R_1], R_2, \ldots)$



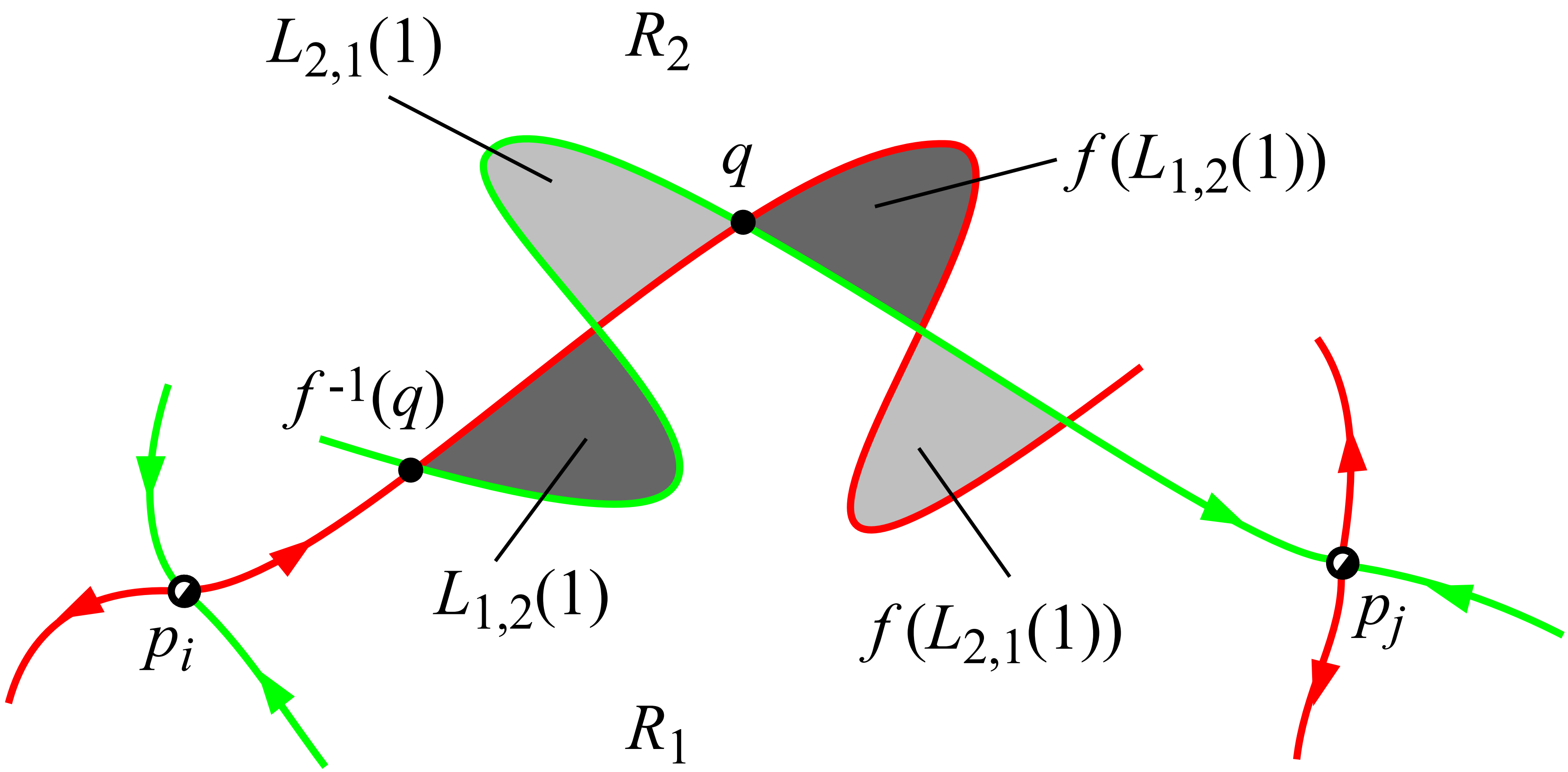
Lobe dynamics: transport across a boundary

- $U[f^{-1}(q), q] \bigcup S[f^{-1}(q), q]$ forms boundary of two lobes; one in R_1, labeled $L_{1,2}(1)$, or equivalently $([R_1], R_2)$, where $f(([R_1], R_2)) = (R_1, [R_2])$, etc. for $L_{2,1}(1)$



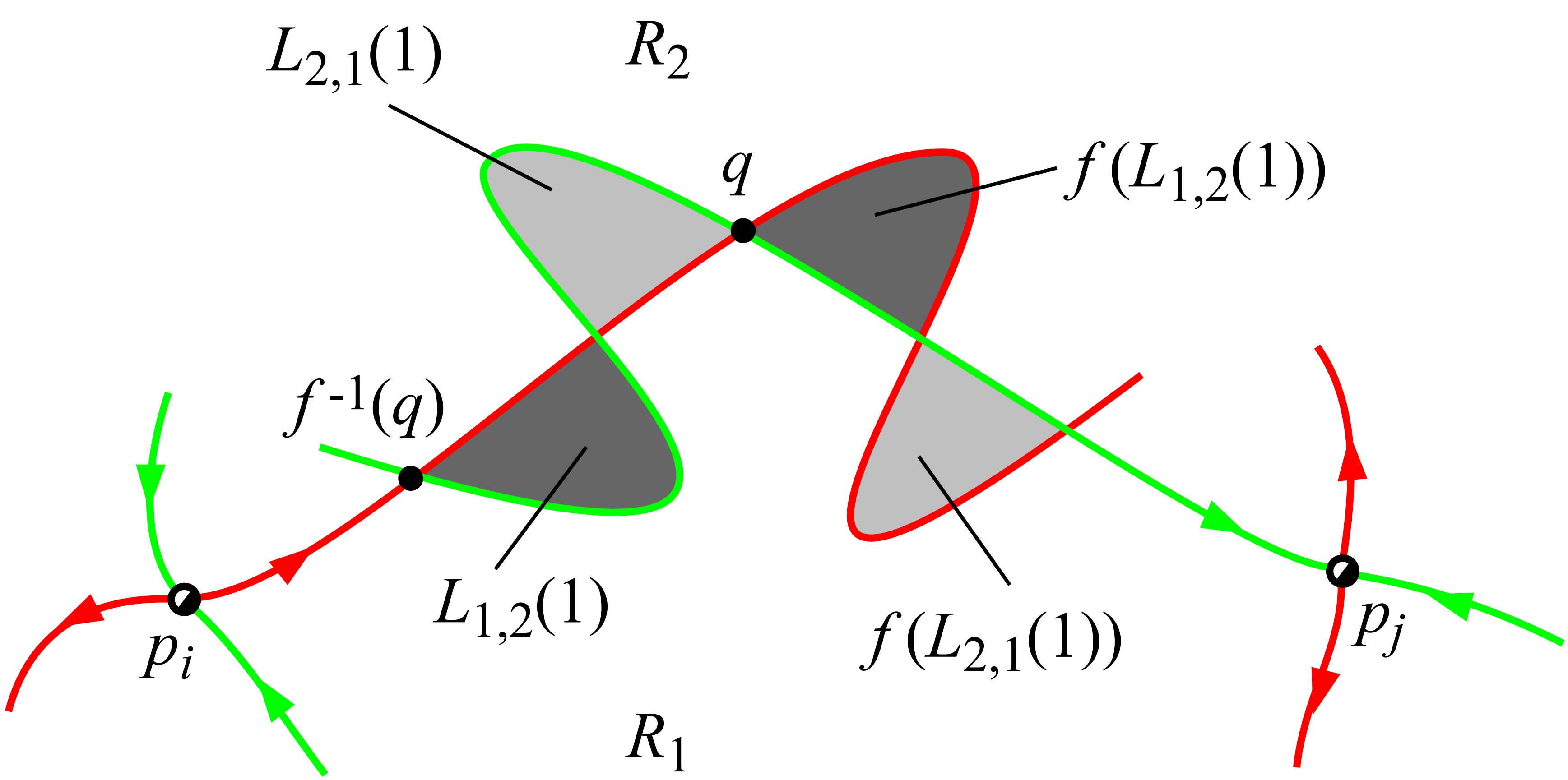
Lobe dynamics: transport across a boundary

- Under one iteration of f, *only points in* $L_{1,2}(1)$ can move from R_1 into R_2 by crossing B, etc.
- The two lobes $L_{1,2}(1)$ and $L_{2,1}(1)$ are called a **turnstile**.



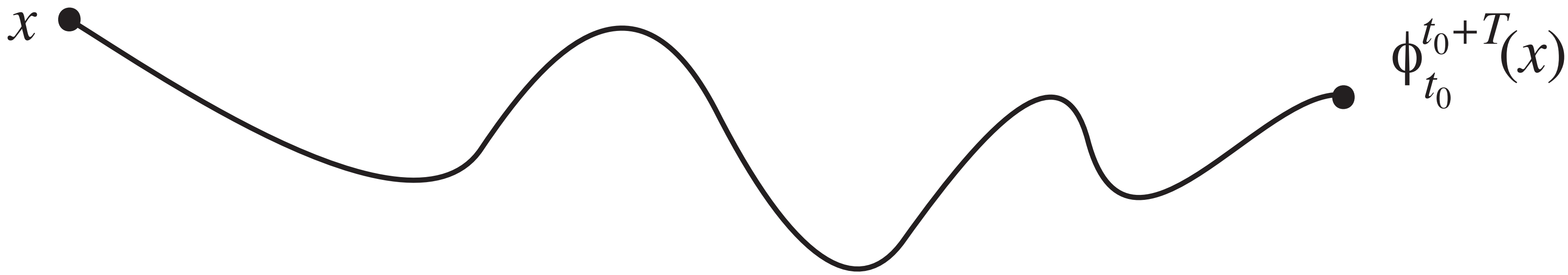
Lobe dynamics: transport across a boundary

- Essence of lobe dynamics: **dynamics associated with crossing a boundary is reduced to the dynamics of turnstile lobes associated with the boundary.**



Extending to realistic flows

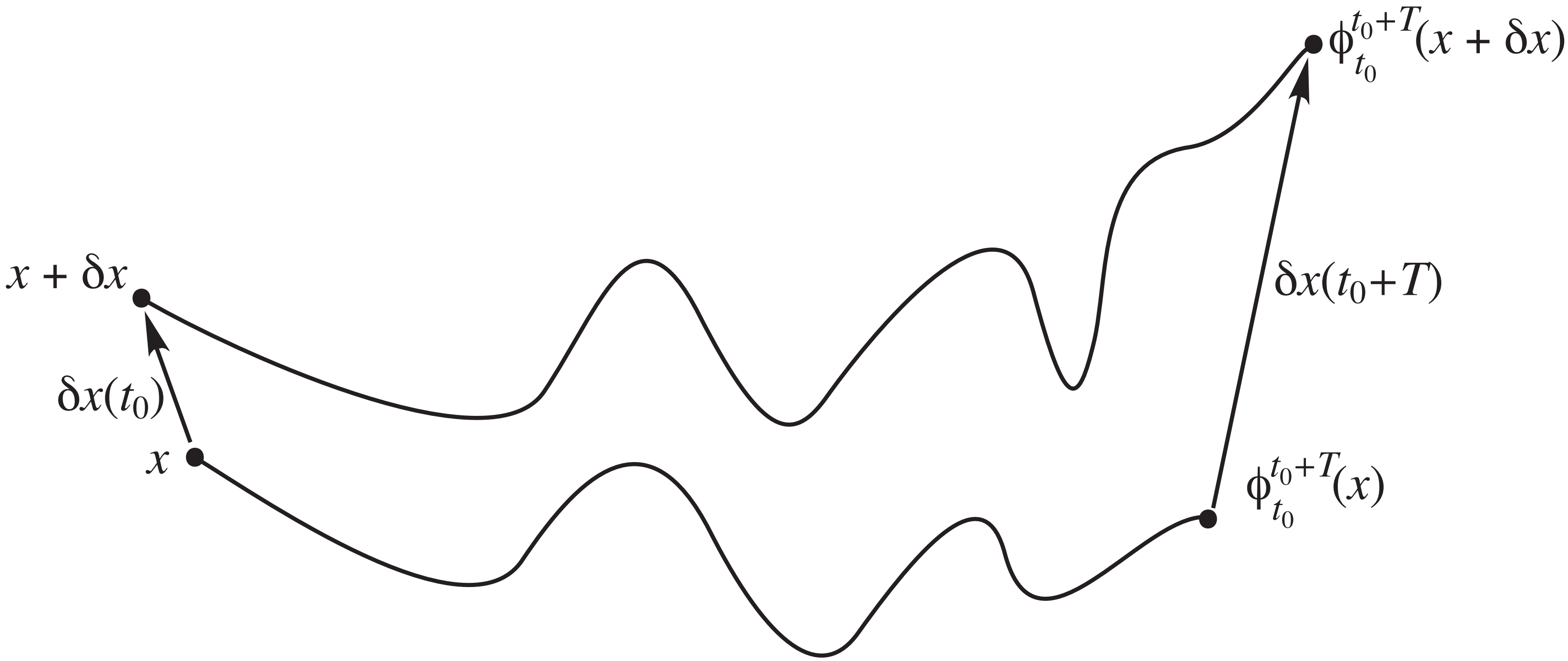
- Data-driven, finite-time, aperiodic setting
- How do we get at transport?
- Recall the flow, $x \mapsto \phi_t^{t+T}(x)$



Identify regions of high sensitivity of initial conditions

- Small initial perturbations $\delta x(t)$ grow like

$$\begin{aligned}\delta x(t+T) &= \phi_t^{t+T}(x+\delta x(t)) - \phi_t^{t+T}(x) \\ &= \frac{d\phi_t^{t+T}(x)}{dx}\delta x(t) + O(||\delta x(t)||^2)\end{aligned}$$



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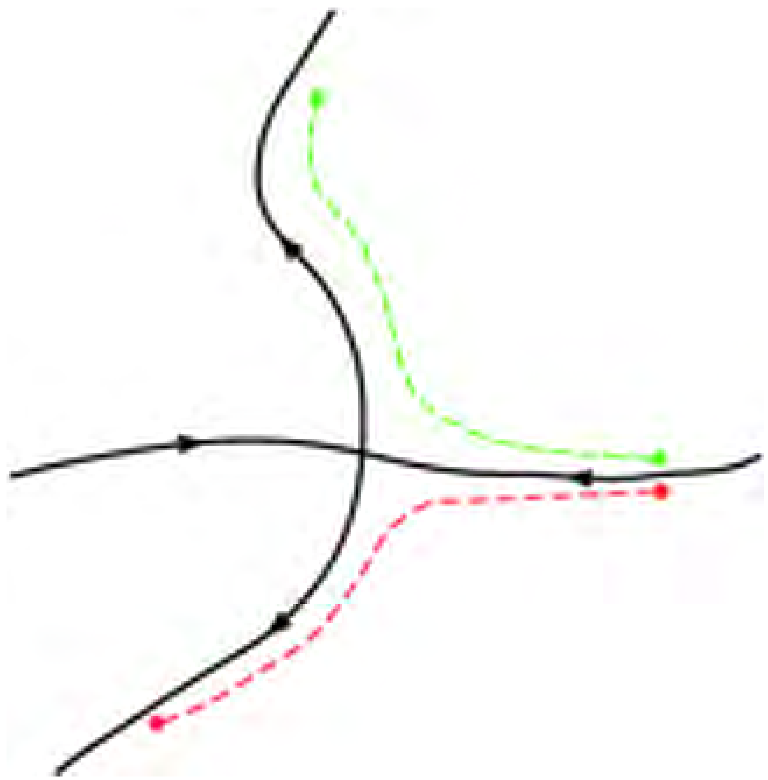


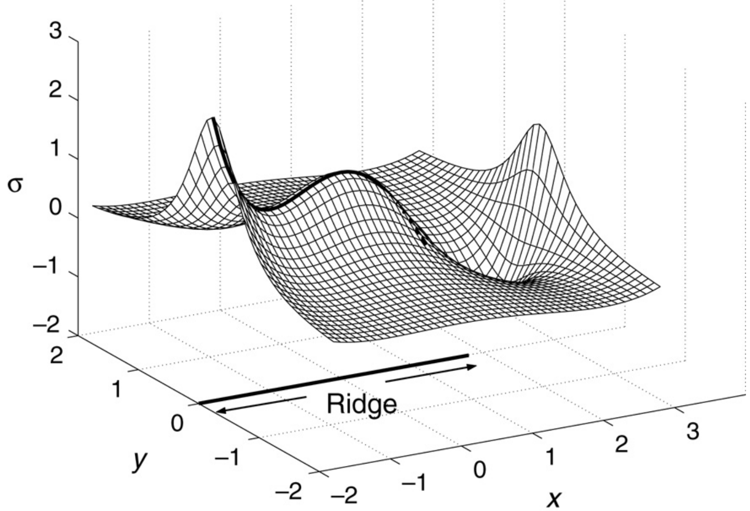
Invariant manifold analogs: FTLE-LCS approach

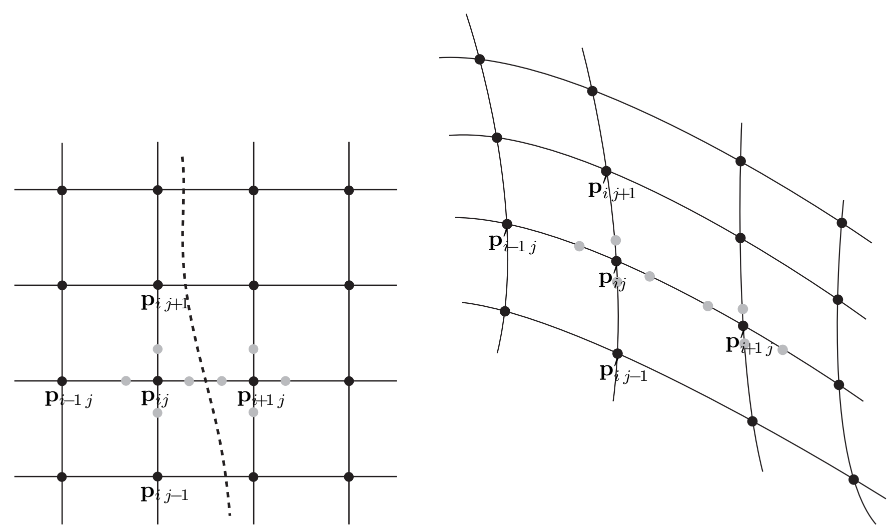
- The finite-time Lyapunov exponent (FTLE),

$$\sigma_t^T(x) = \frac{1}{|T|} \log \left\| \frac{d\phi_t^{t+T}(x)}{dx} \right\|$$

measures the maximum stretching rate over the interval T of trajectories starting near the point x at time t

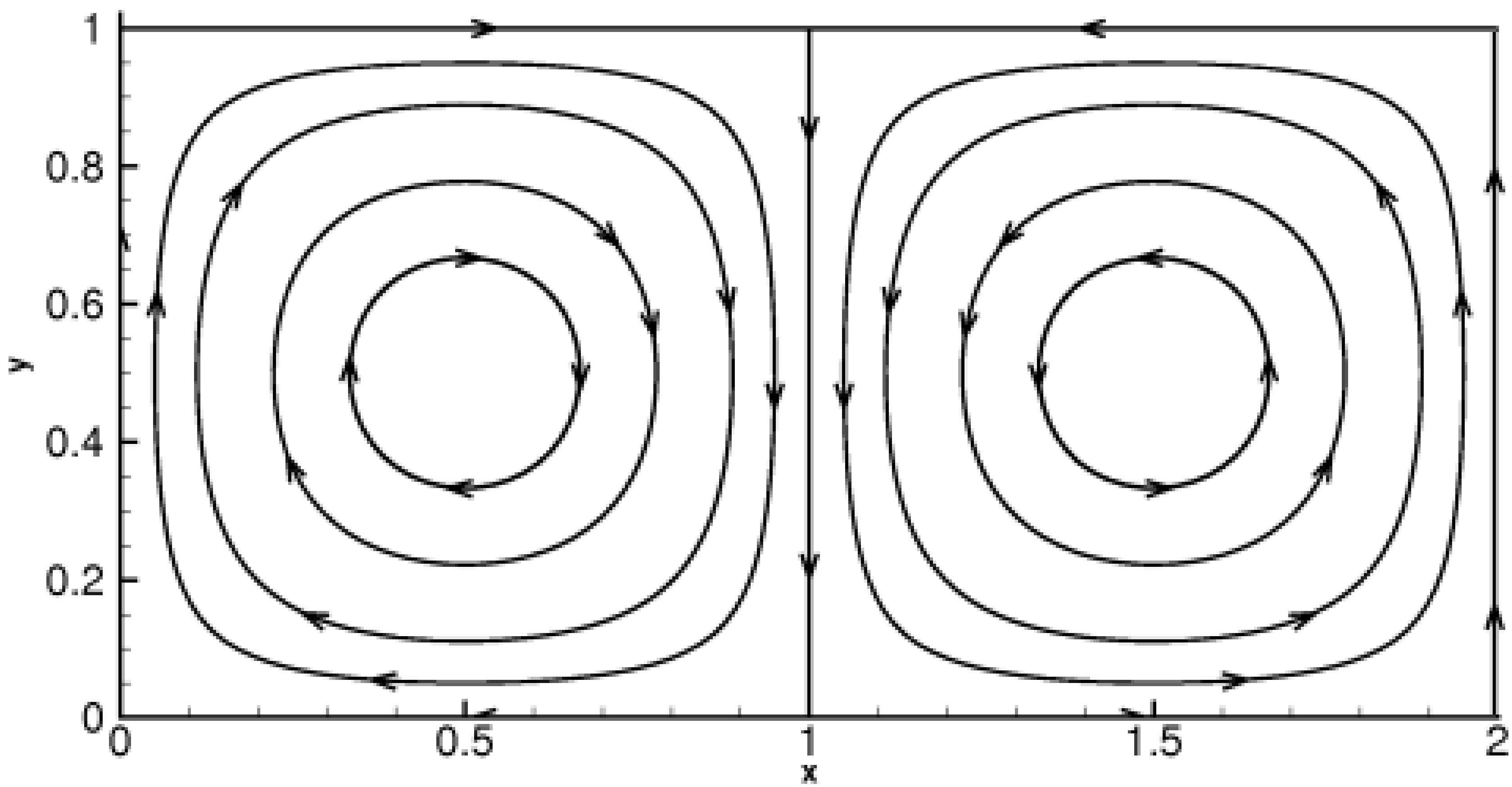
- Ridges of σ_t^T are candidate hyperbolic codim-1 surfaces; finite-time analogs of stable/unstable manifolds; Lagrangian coherent structures[1]



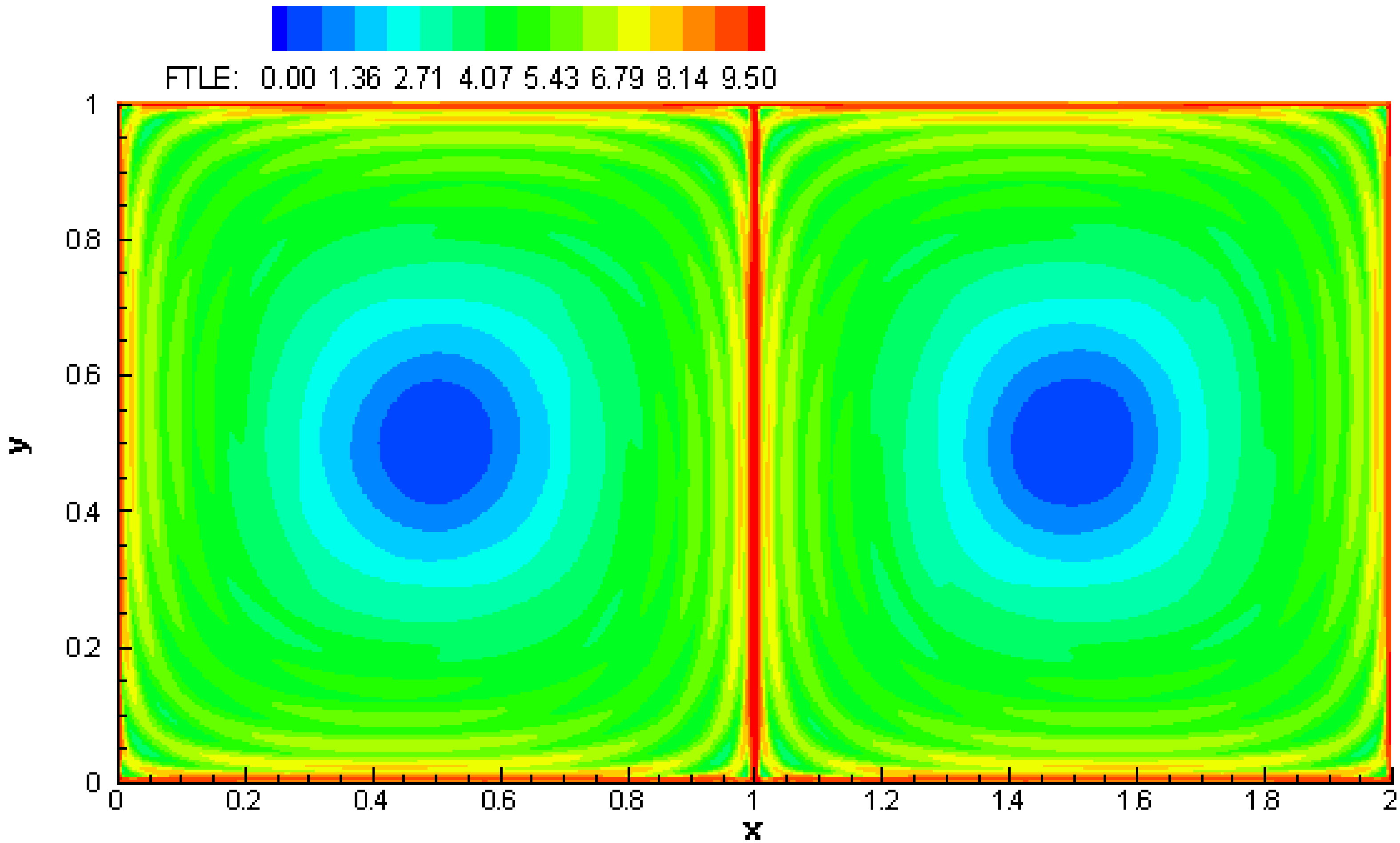


[1] cf. Bowman, 1999; Haller & Yuan, 2000; Haller, 2001; Shadden, Lekien, Marsden, 2005

Invariant manifold analogs: FTLE-LCS approach



Invariant manifold analogs: FTLE-LCS approach

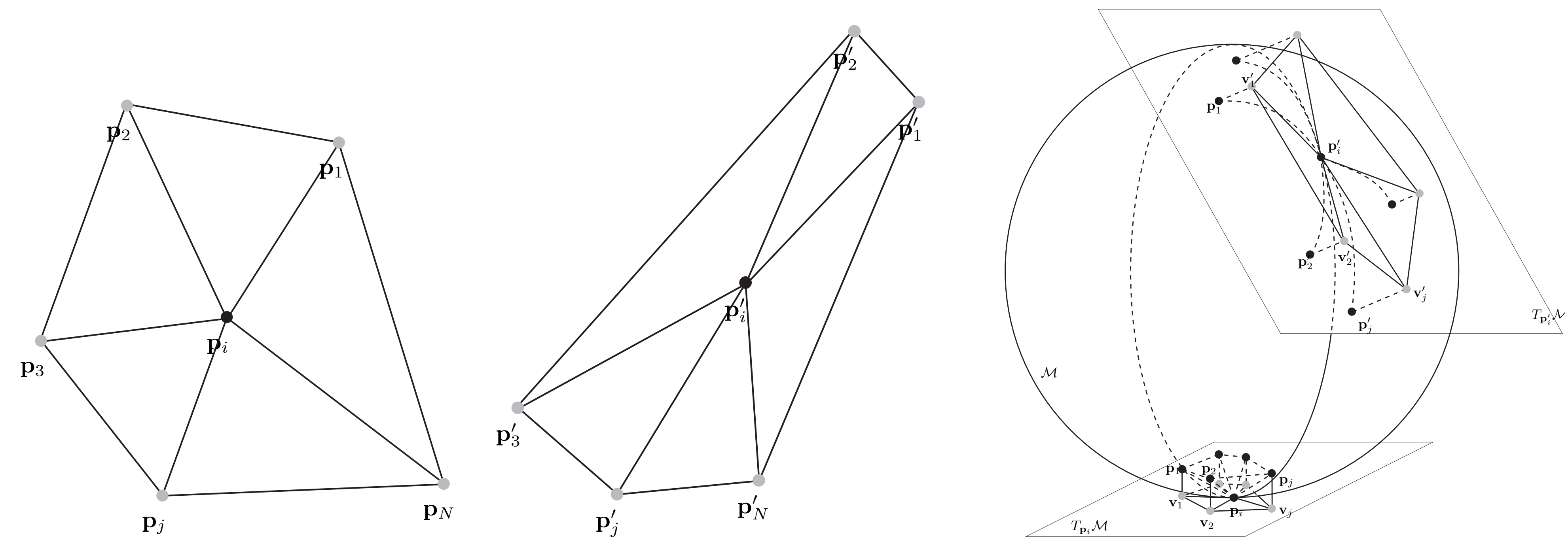


Invariant manifold analogs: FTLE-LCS approach

- We can define the FTLE for Riemannian manifolds[2]

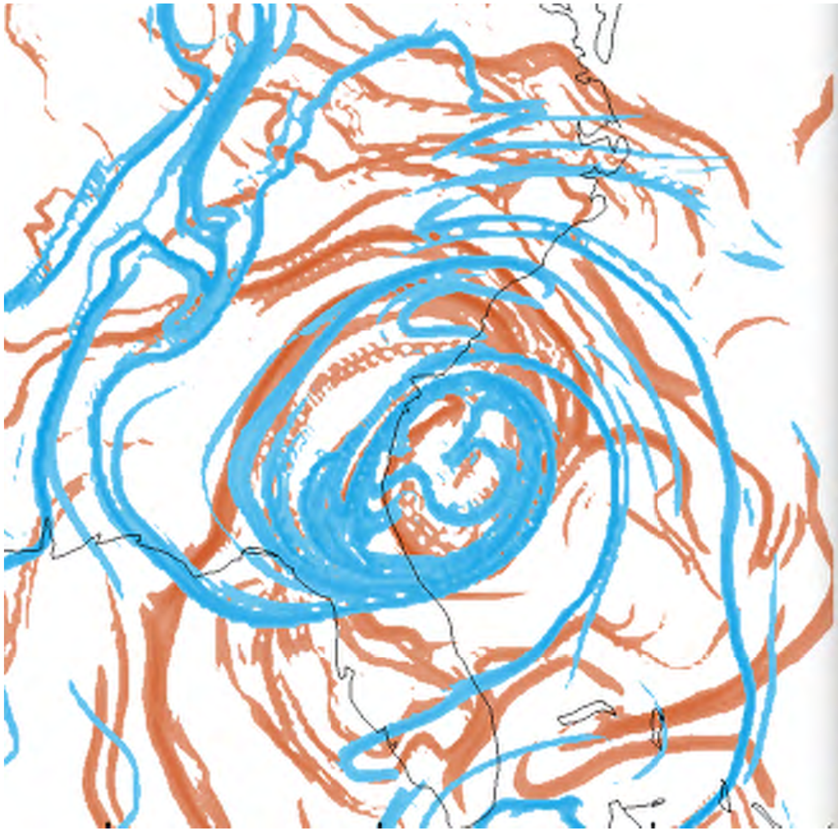
$$\sigma_t^T(x) = \frac{1}{|T|} \ln \left\| \mathrm{D}\phi_t^{t+T} \right\| \doteq \frac{1}{|T|} \ln \left(\max_{\mathrm{y} \neq 0} \frac{\left\| \mathrm{D}\phi_t^{t+T}(\mathrm{y}) \right\|}{\|\mathrm{y}\|} \right)$$

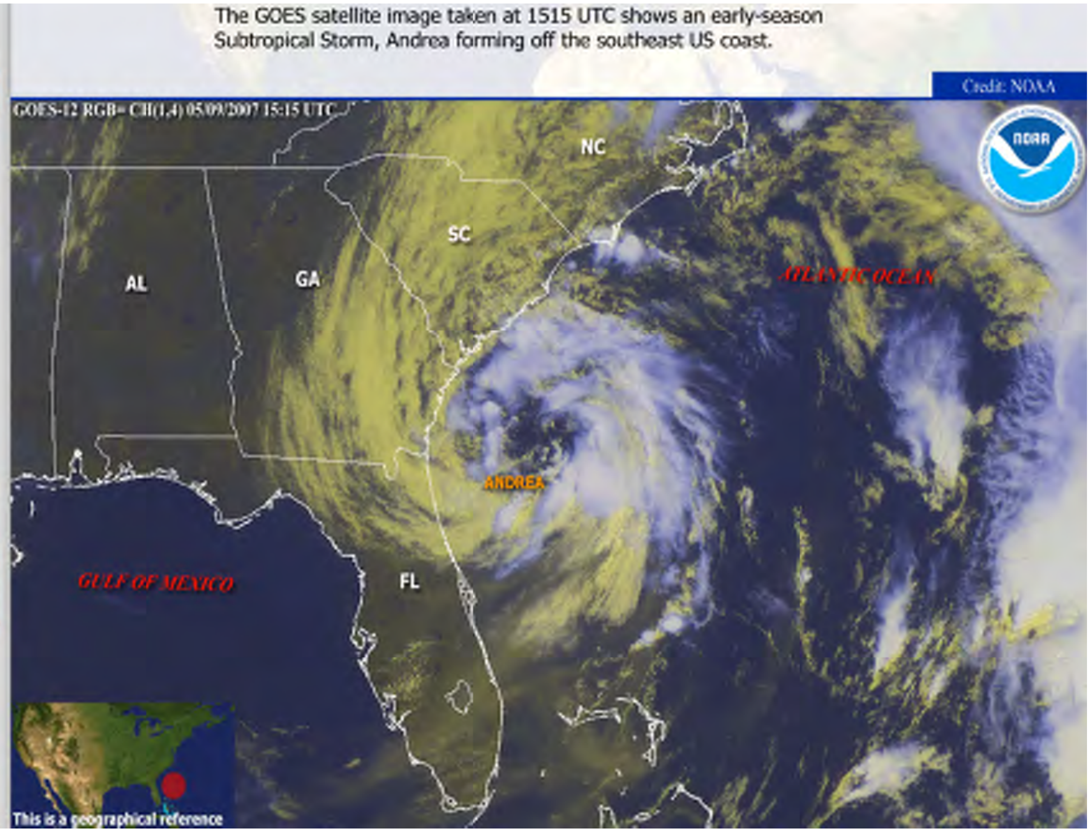
with y a small perturbation in the tangent space at x.



[2] Lekien & Ross [2010] Chaos

Hurricanes and lobe dynamics





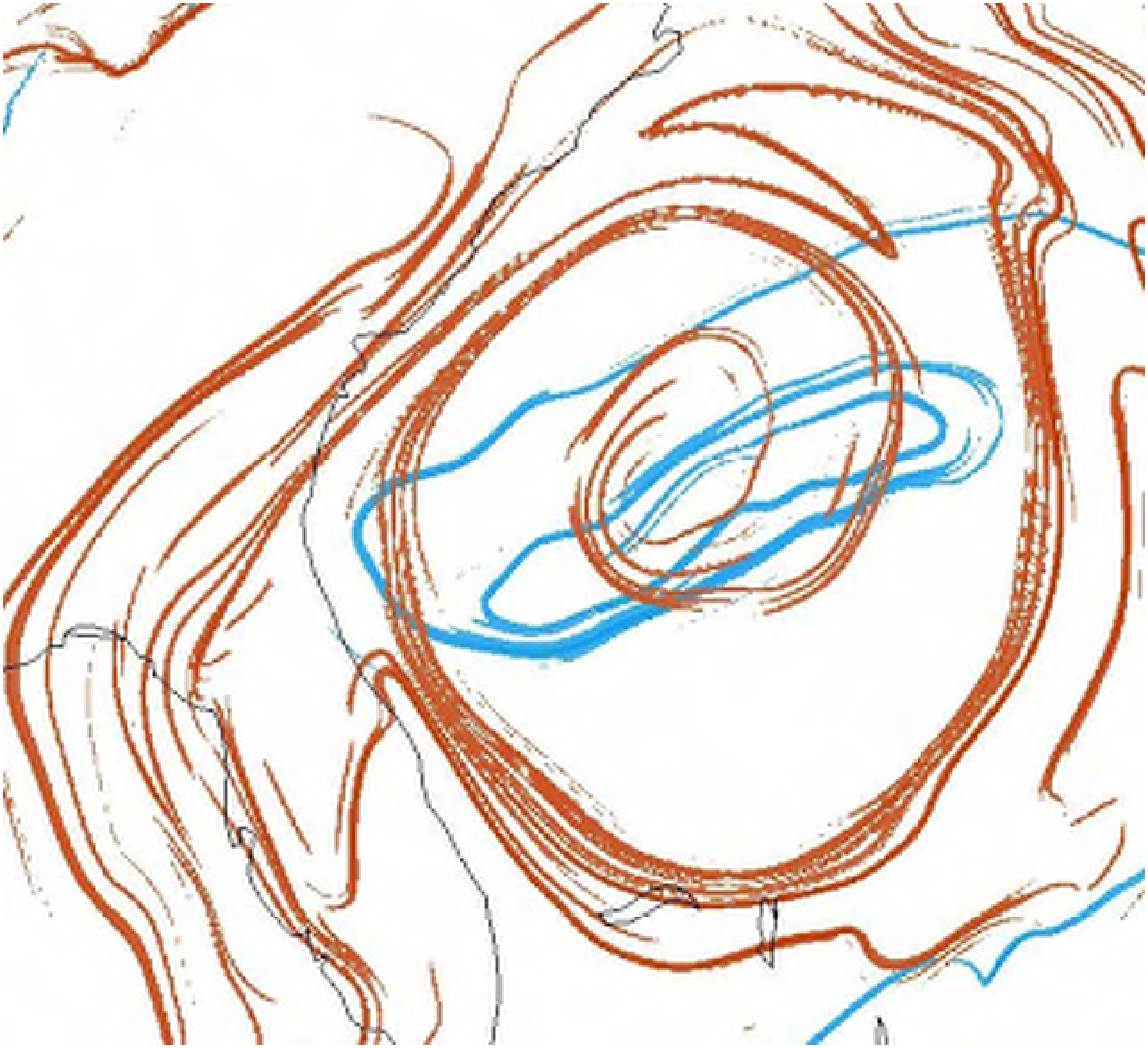
orange = repelling LCSs, blue = attracting LCSs

satellite

Andrea, first storm of 2007 hurricane season

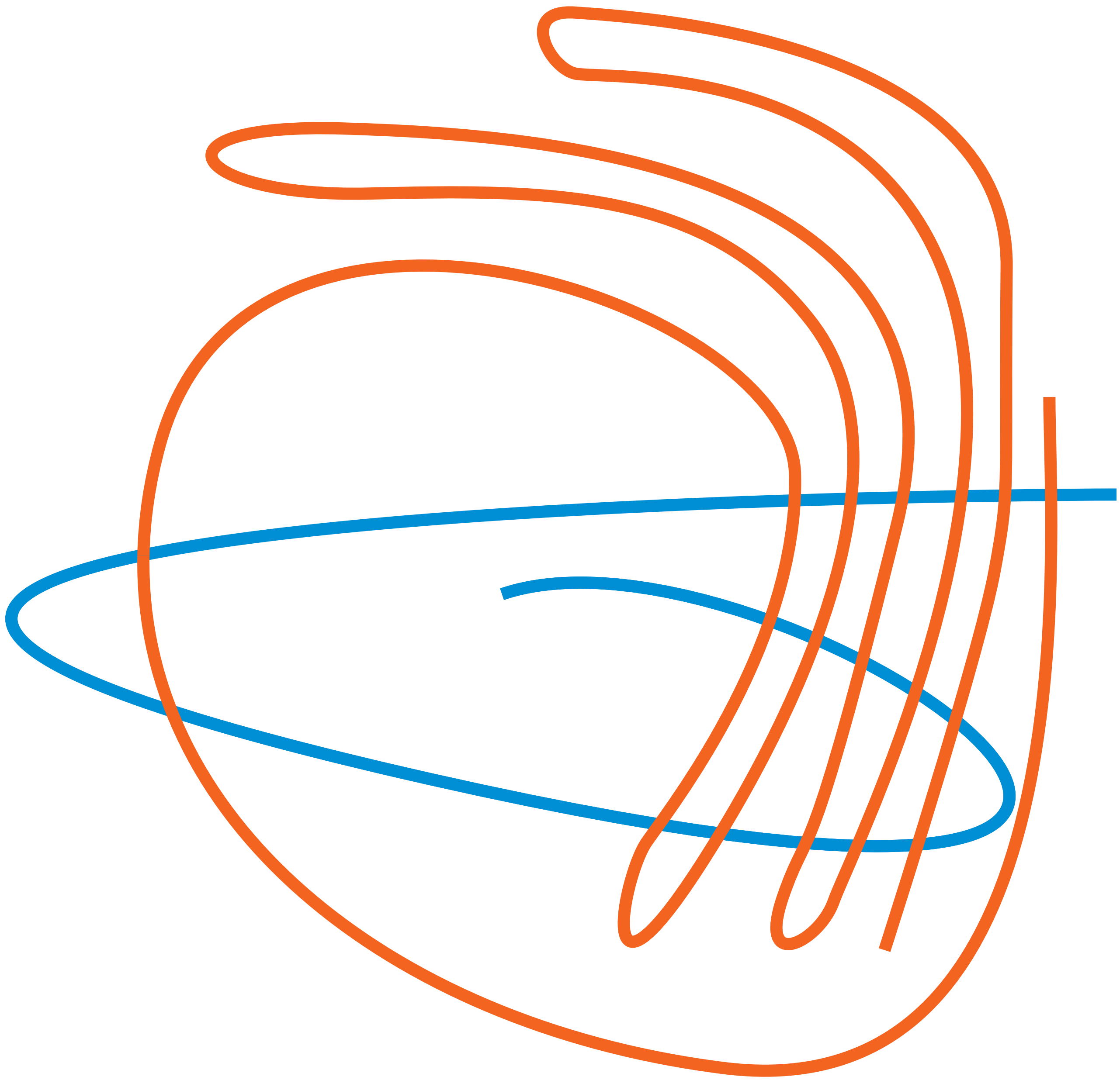
cf. Sapsis & Haller [2009], Du Toit & Marsden [2010], Lekien & Ross [2010], Tallapragada & Ross [2011]

Hurricanes and lobe dynamics



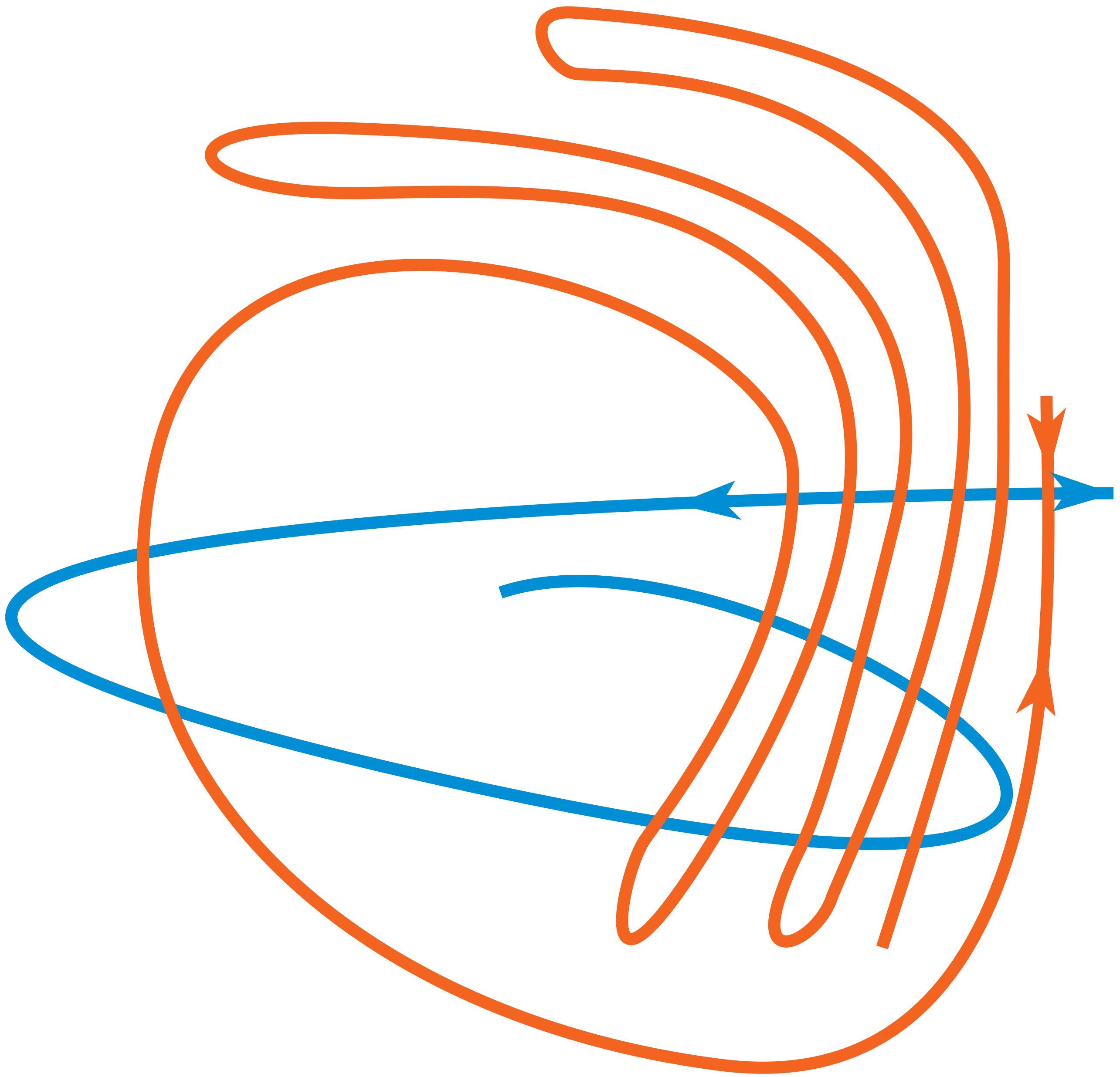
Andrea at one snapshot; LCS shown (orange = repelling, blue = attracting)

Hurricanes and lobe dynamics



orange = repelling (stable manifold), blue = attracting (unstable manifold)

Hurricanes and lobe dynamics



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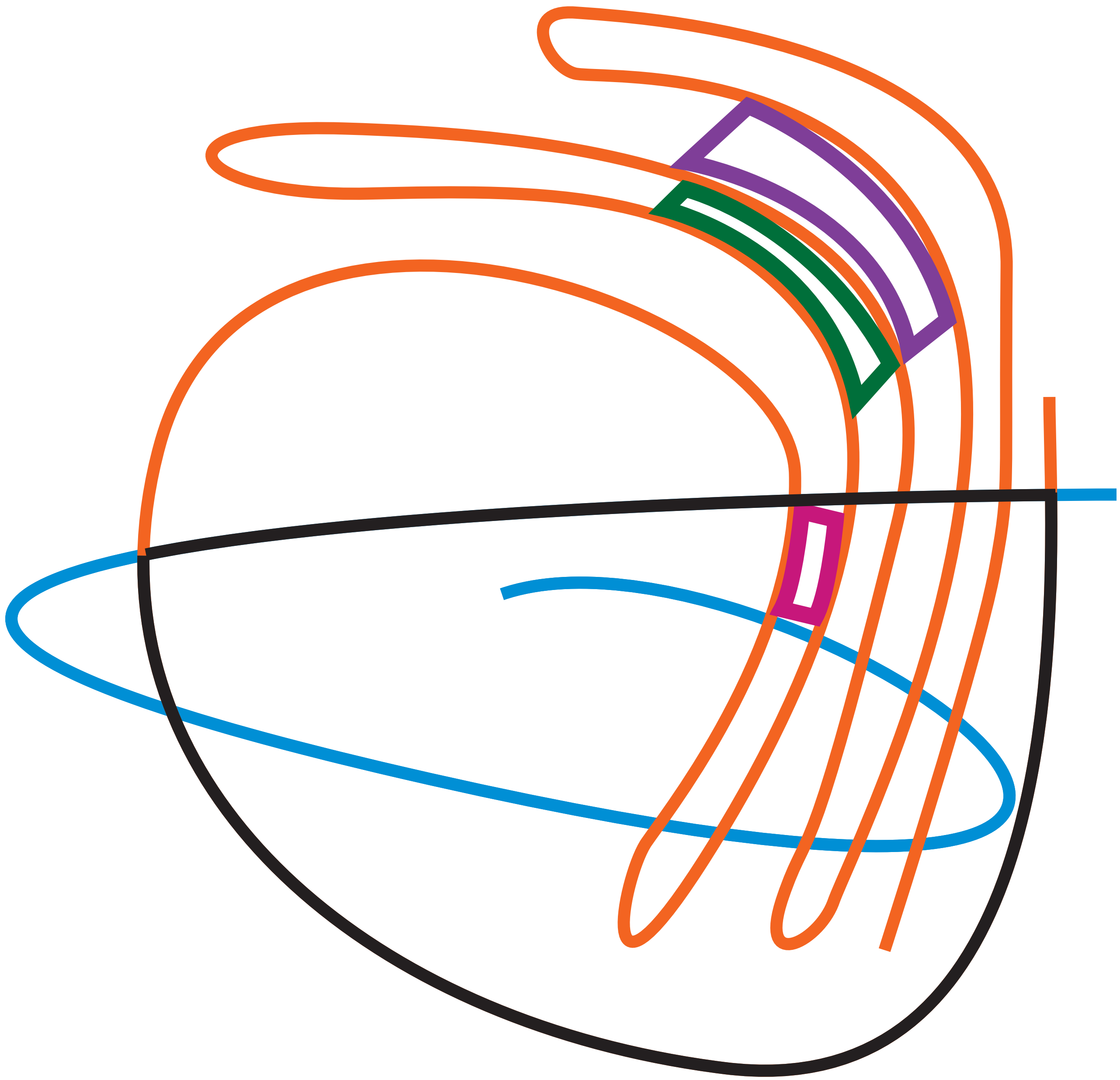
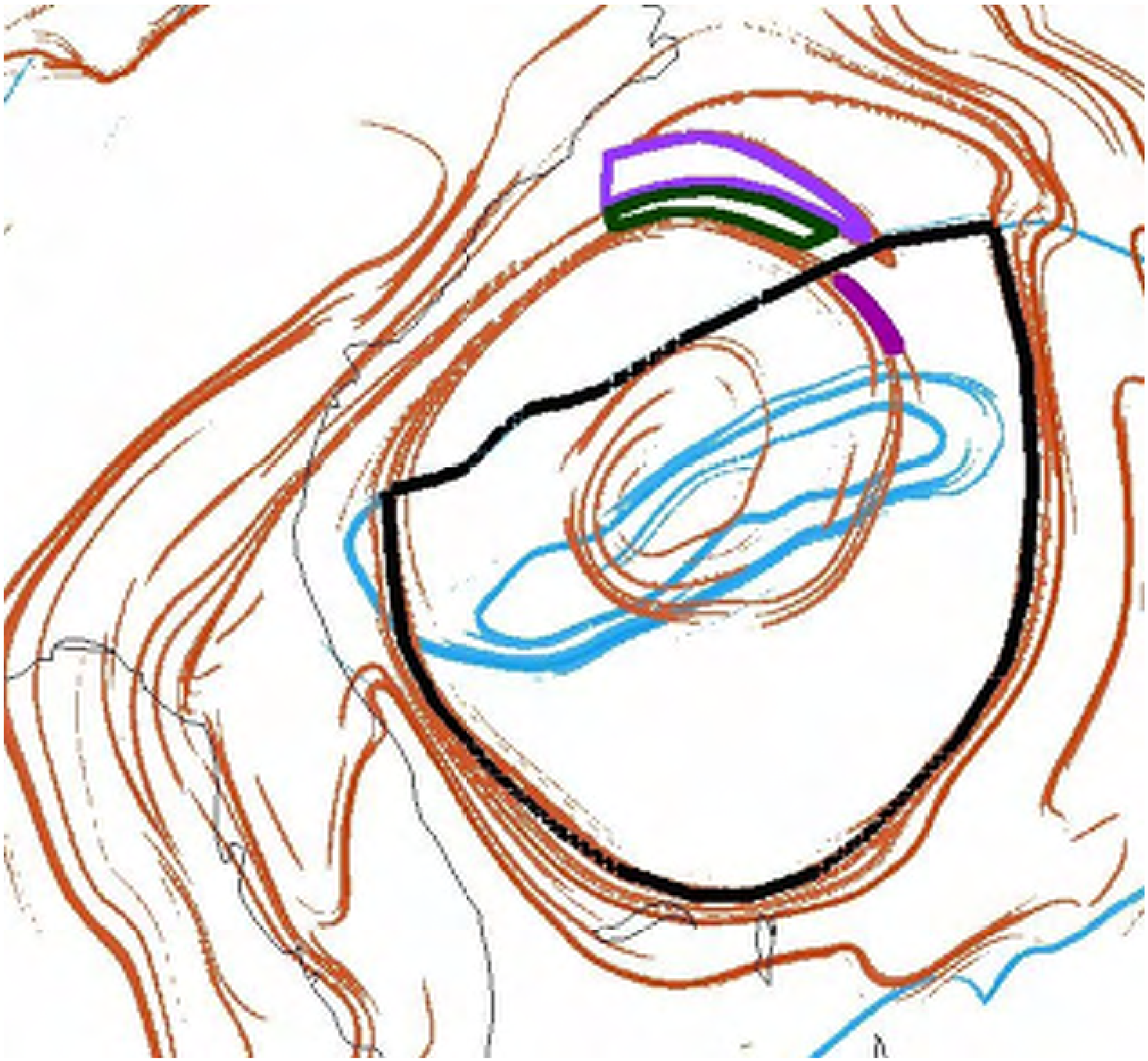
Hurricanes and lobe dynamics



Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out

Hurricanes and lobe dynamics

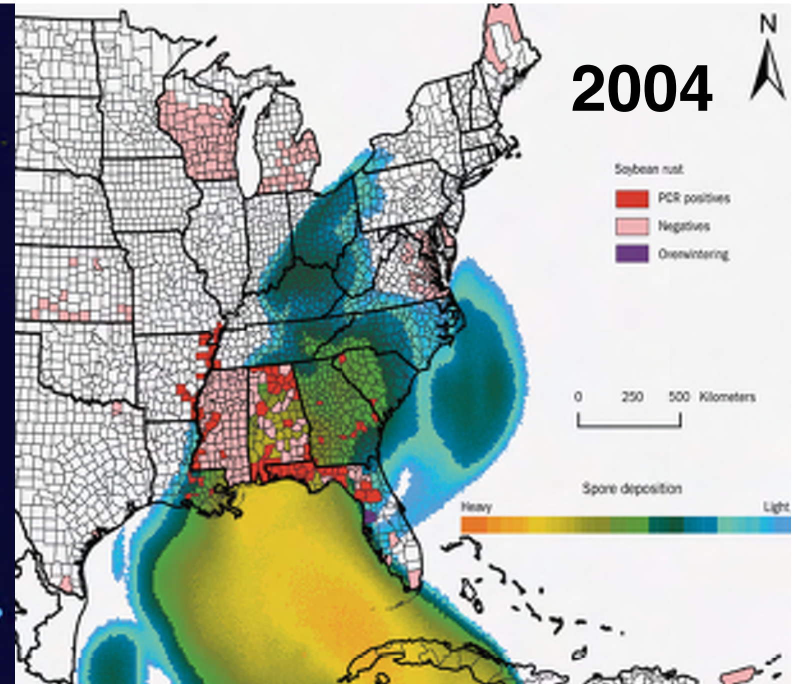


Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out

Hurricanes and lobe dynamics

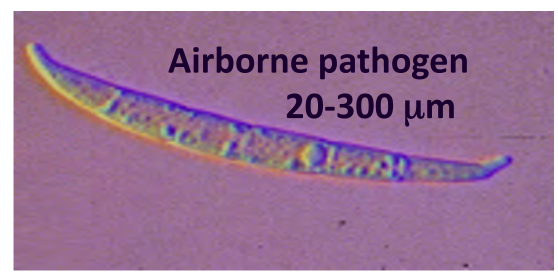
Sets behave as lobe dynamics dictates

Invasive species riding coherent structures

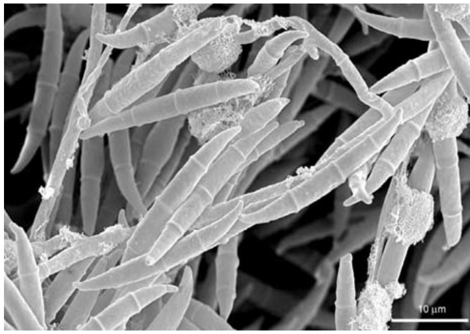


Disease extent

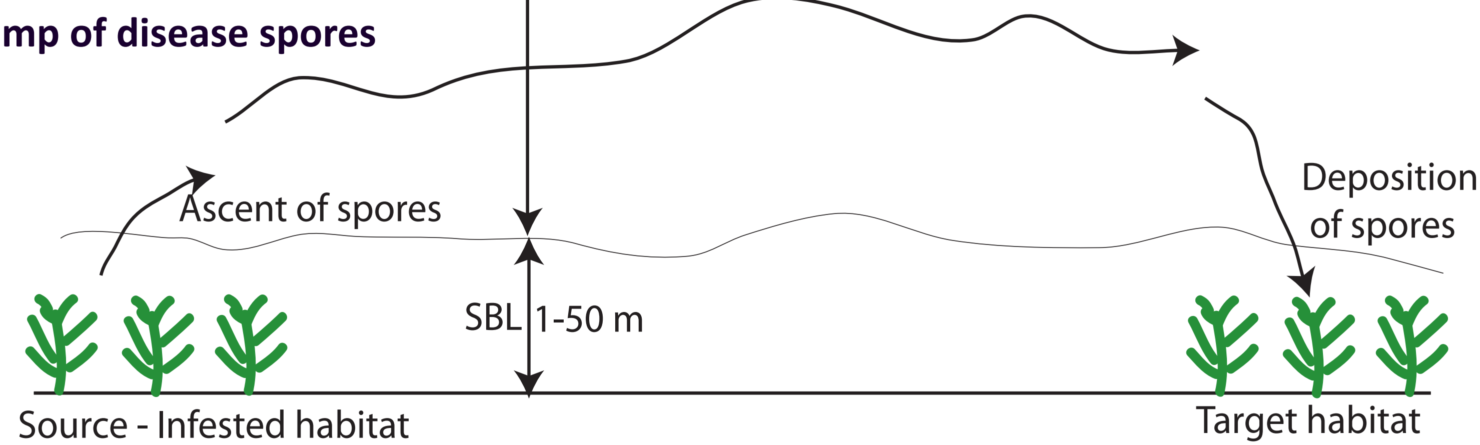
Cost of invasive organisms is ***<u>$137 billion</u>*** **per year in U.S.**



Atmospheric transport of microorganisms



Clump of disease spores



e.g., *Fusarium*

- **Spore production, release, escape from surface**
- **Long-range transport (time-scale hours to days)**
- **Deposition, infection efficiency, host susceptibility**

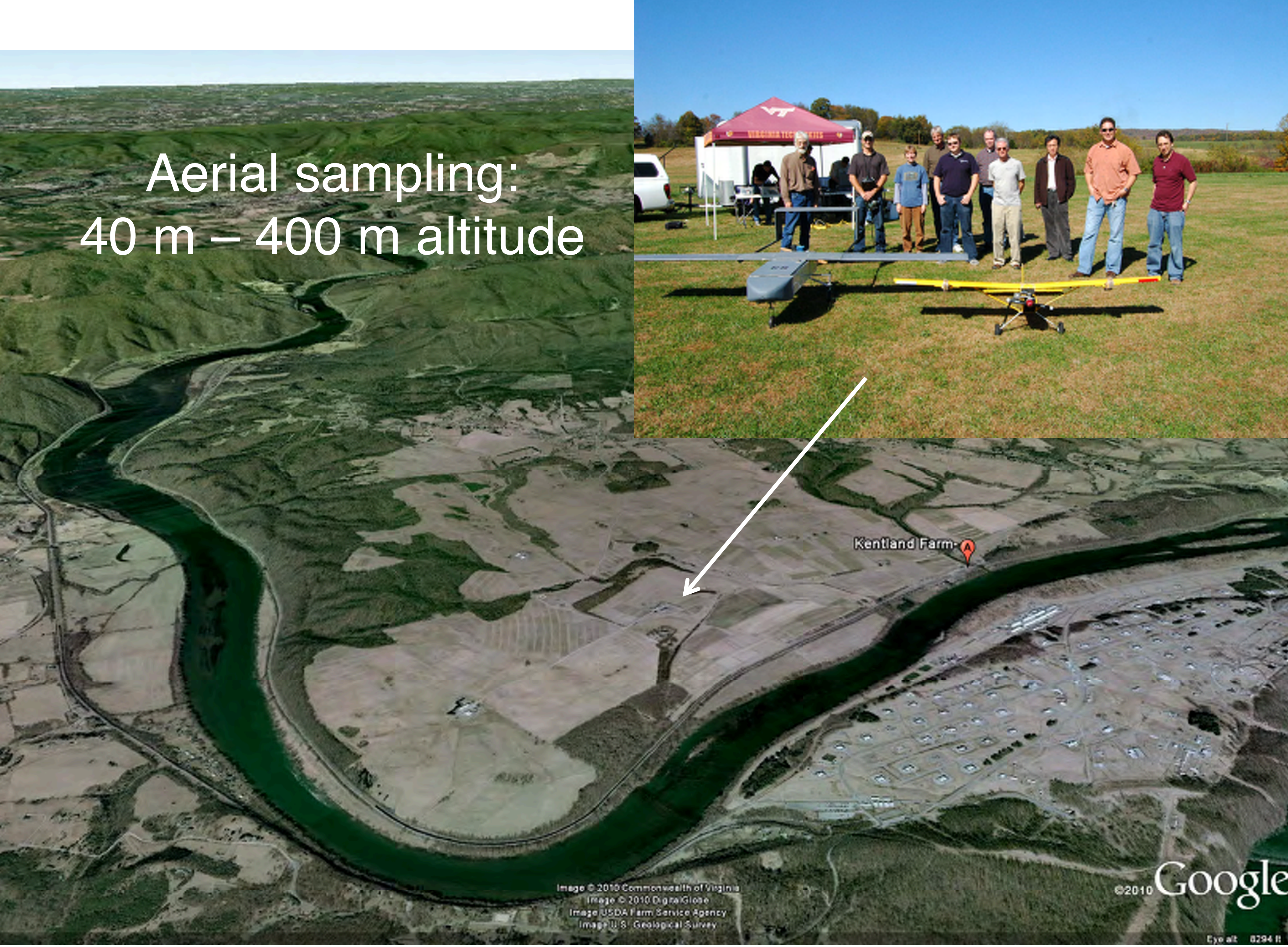
Isard & Gage [2001]

Atmospheric transport network relevant for aeroecology

Skeleton of large-scale horizontal transport

relevant for large-scale spatiotemporal patterns of important biota e.g., plant pathogens

orange = repelling LCSs, blue = attracting LCSs


Aerial sampling:
40 m – 400 m altitude
Kentland Farm
Image © 2010 Commonwealth of Virginia
Image © 2010 DigitalGlobe
Image USDA Farm Service Agency
Image U.S. Geological Survey
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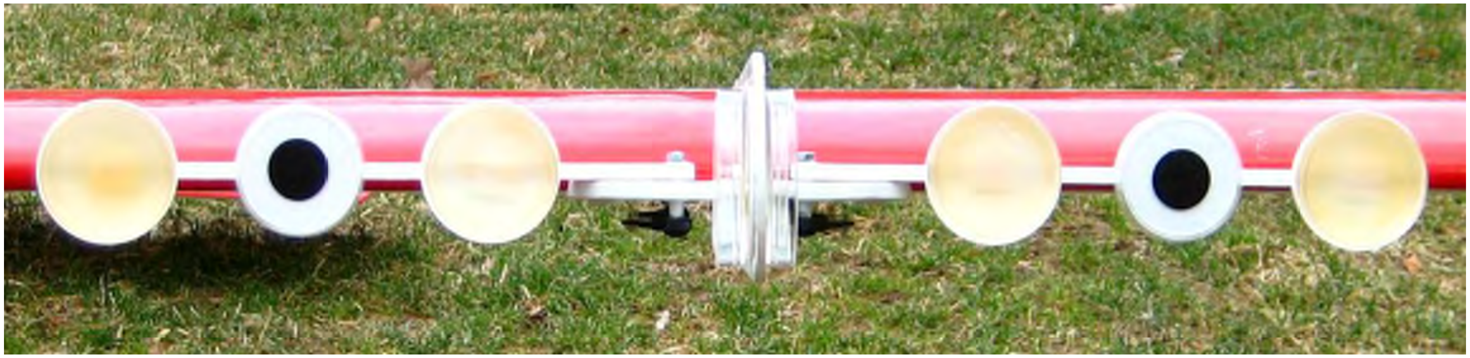

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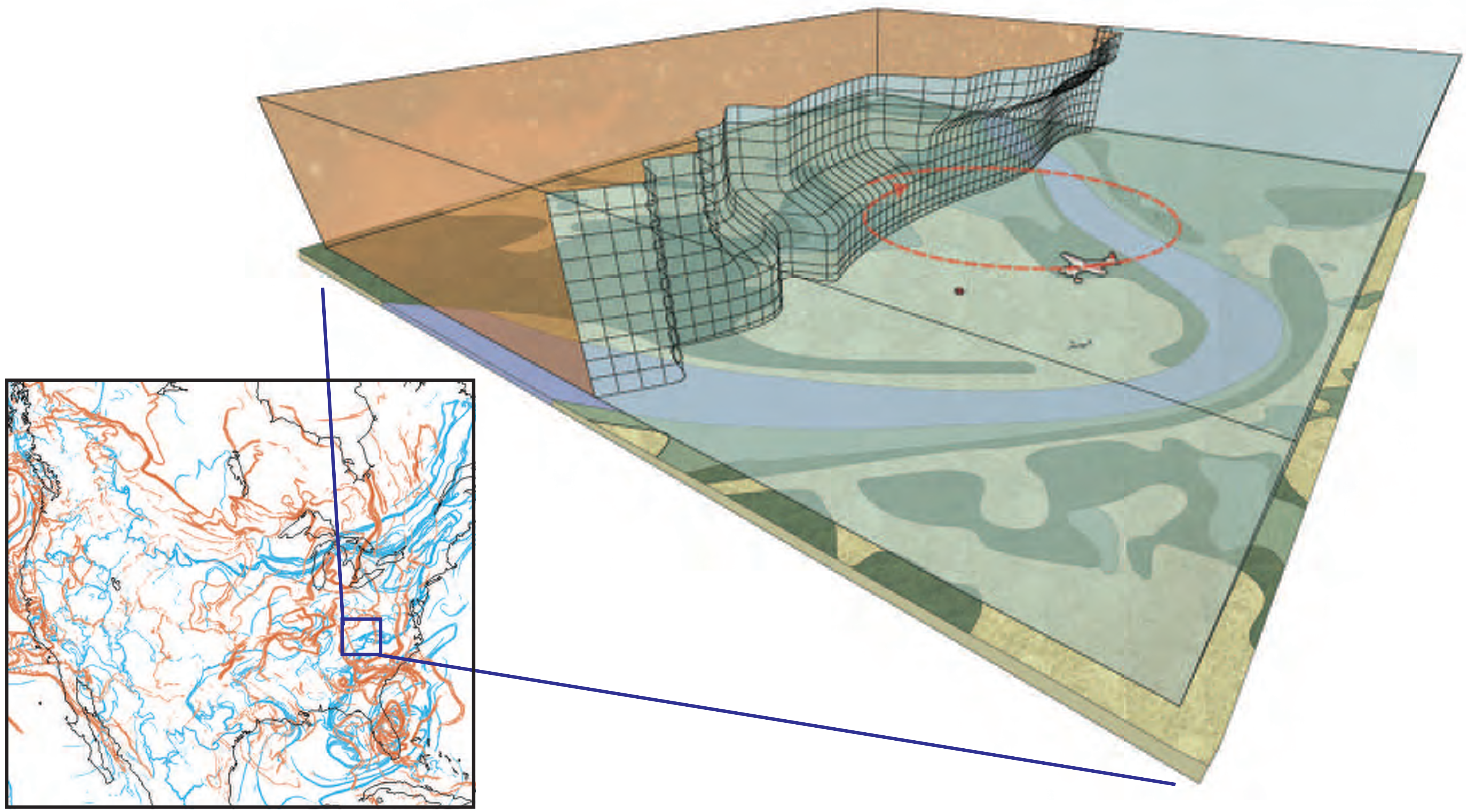
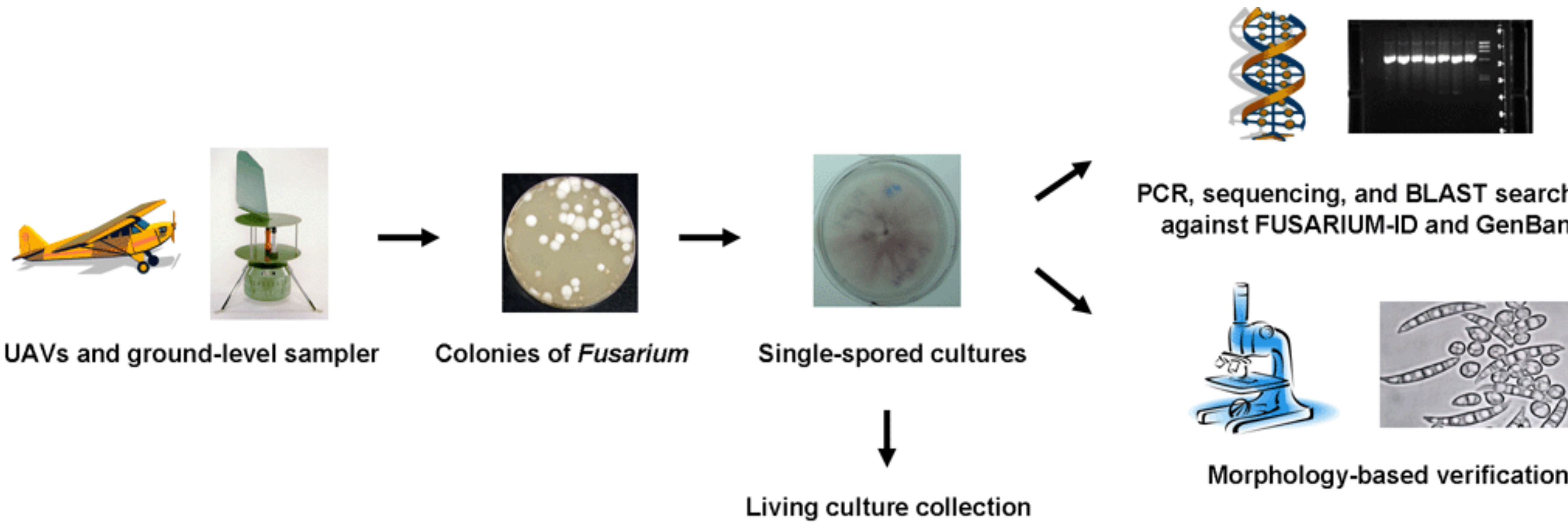
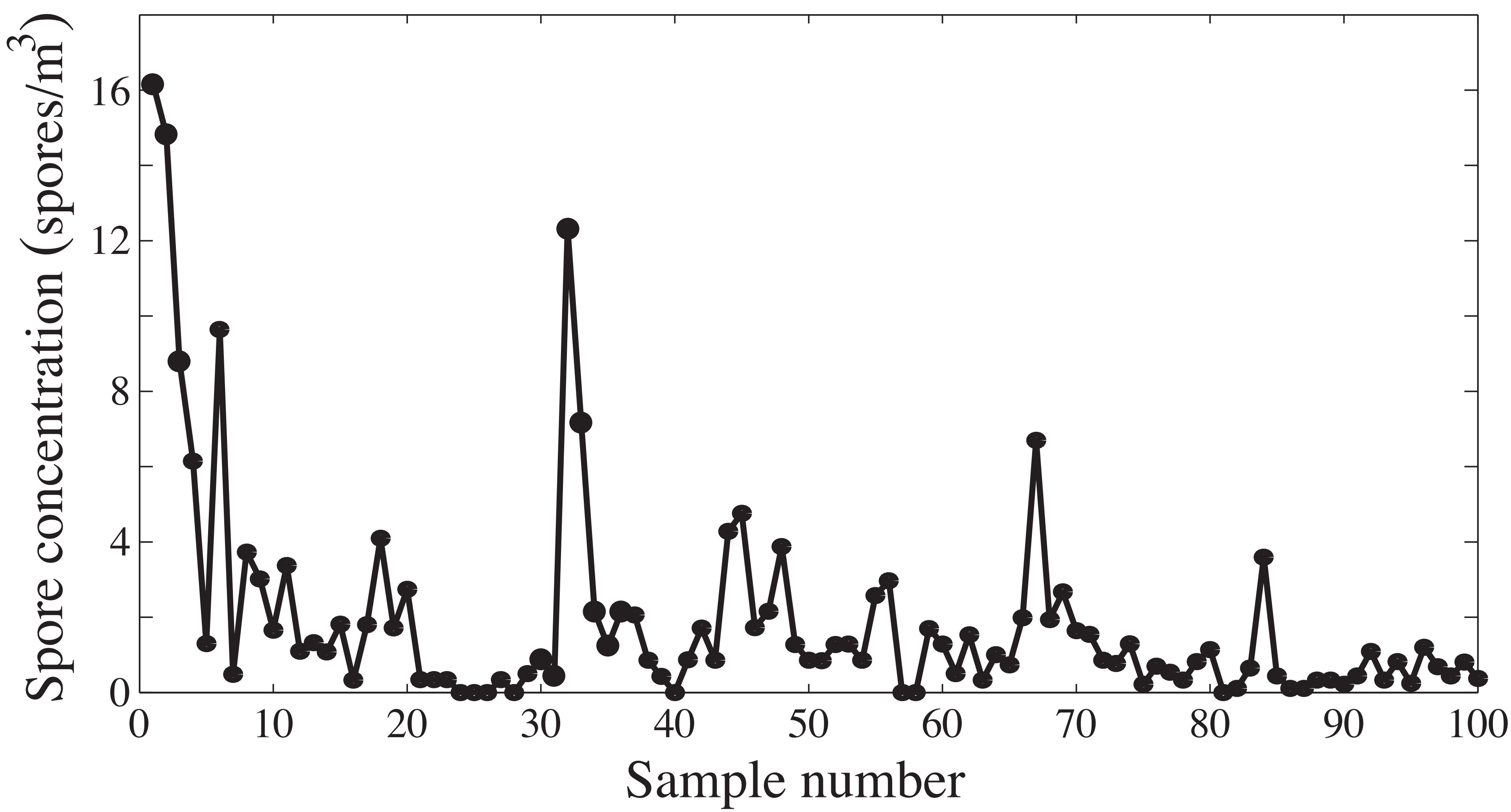
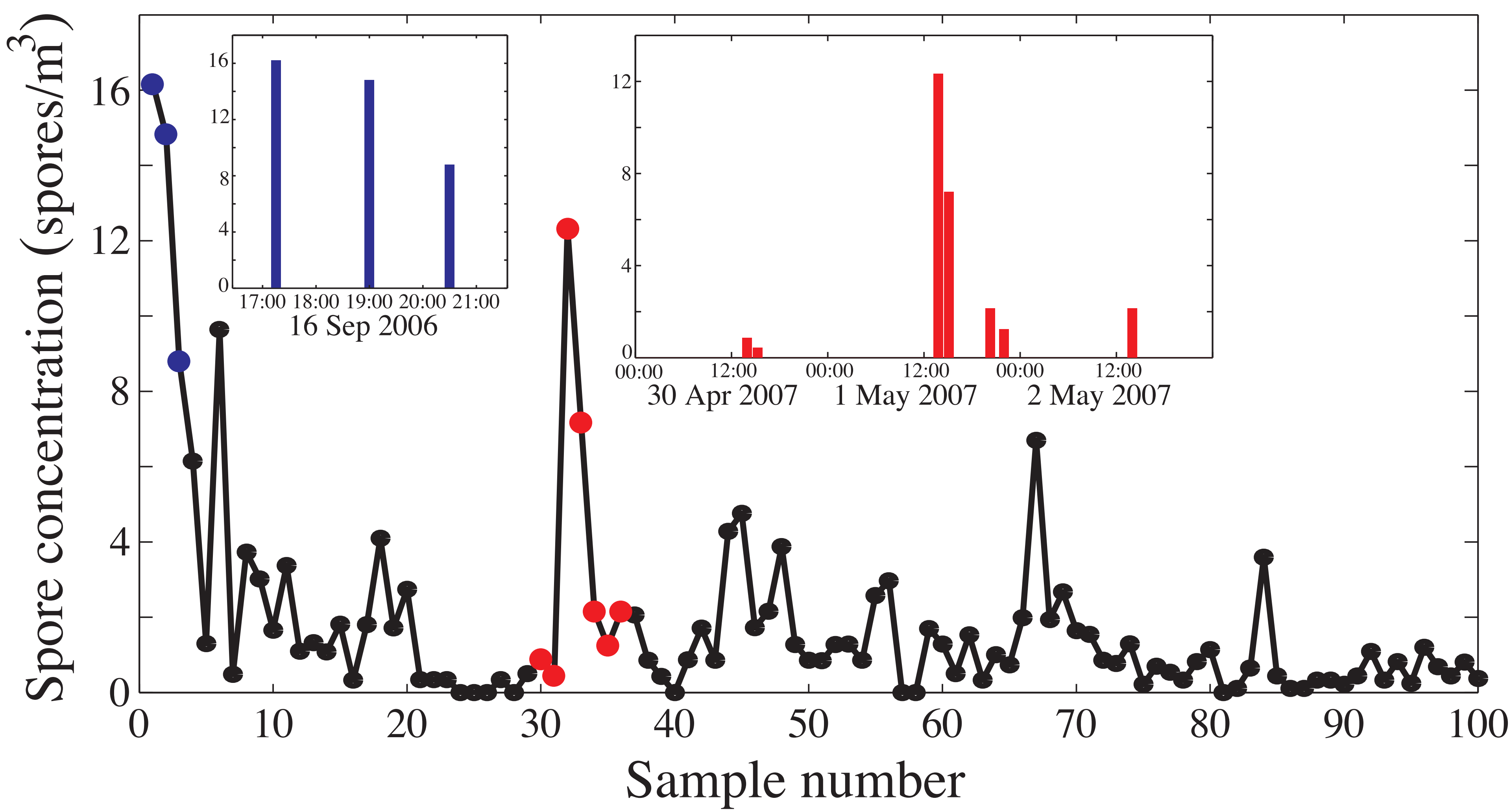



UAVs and ground-level sampler
Colonies of *Fusarium*
Single-spored cultures
Living culture collection
PCR, sequencing, and BLAST searches against FUSARIUM-ID and GenBank
Morphology-based verification



Concentration of *Fusarium* spores (number/m^3) for samples from 100 flights conducted between August 2006 and March 2010.



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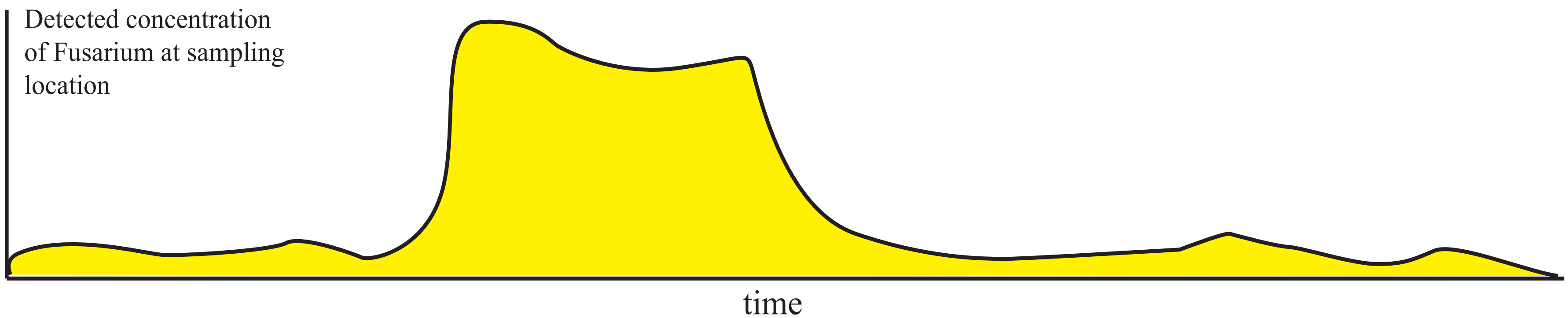
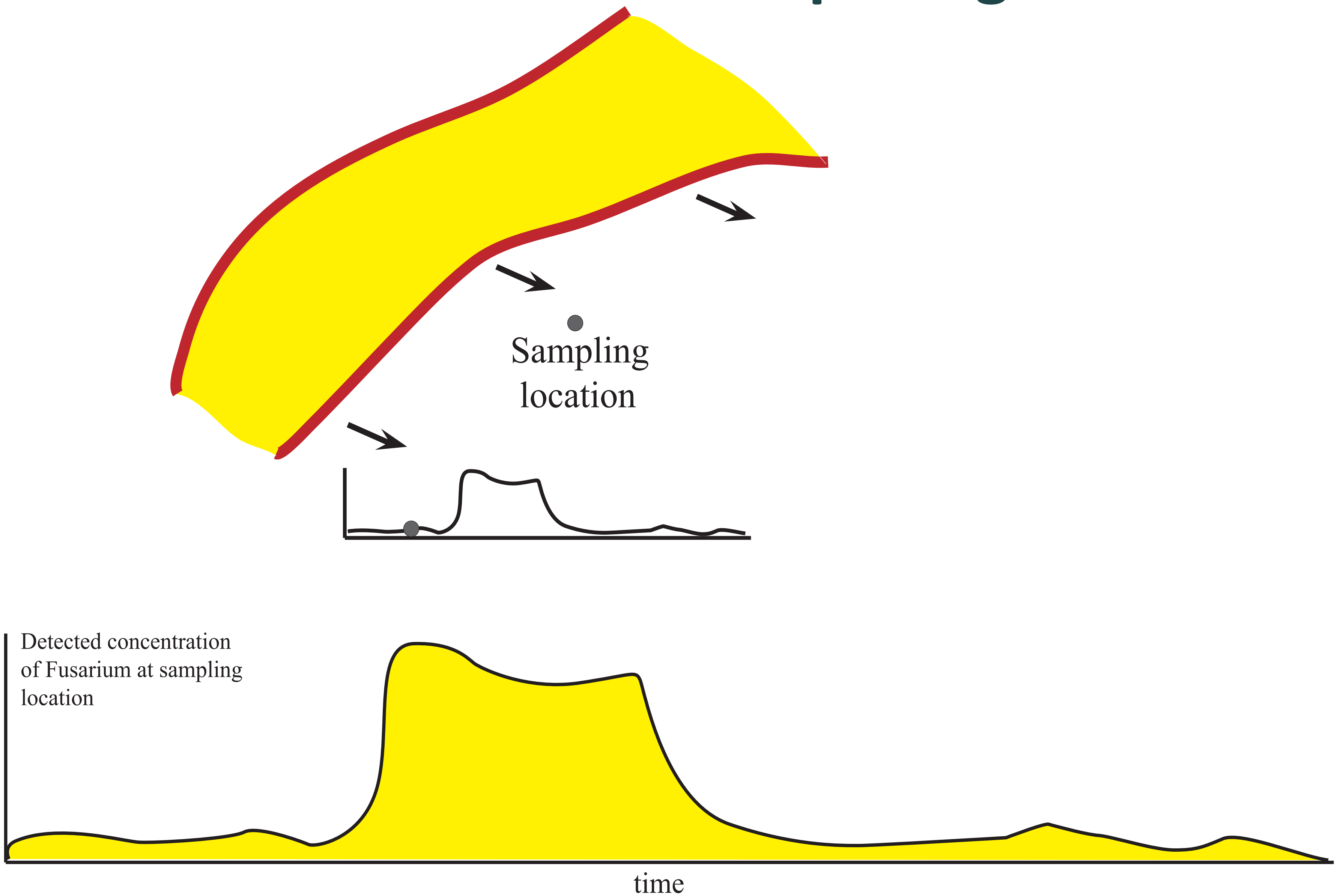
Punctuated changes: correlated to LCS passage?



Punctuated changes: correlated to LCS passage?



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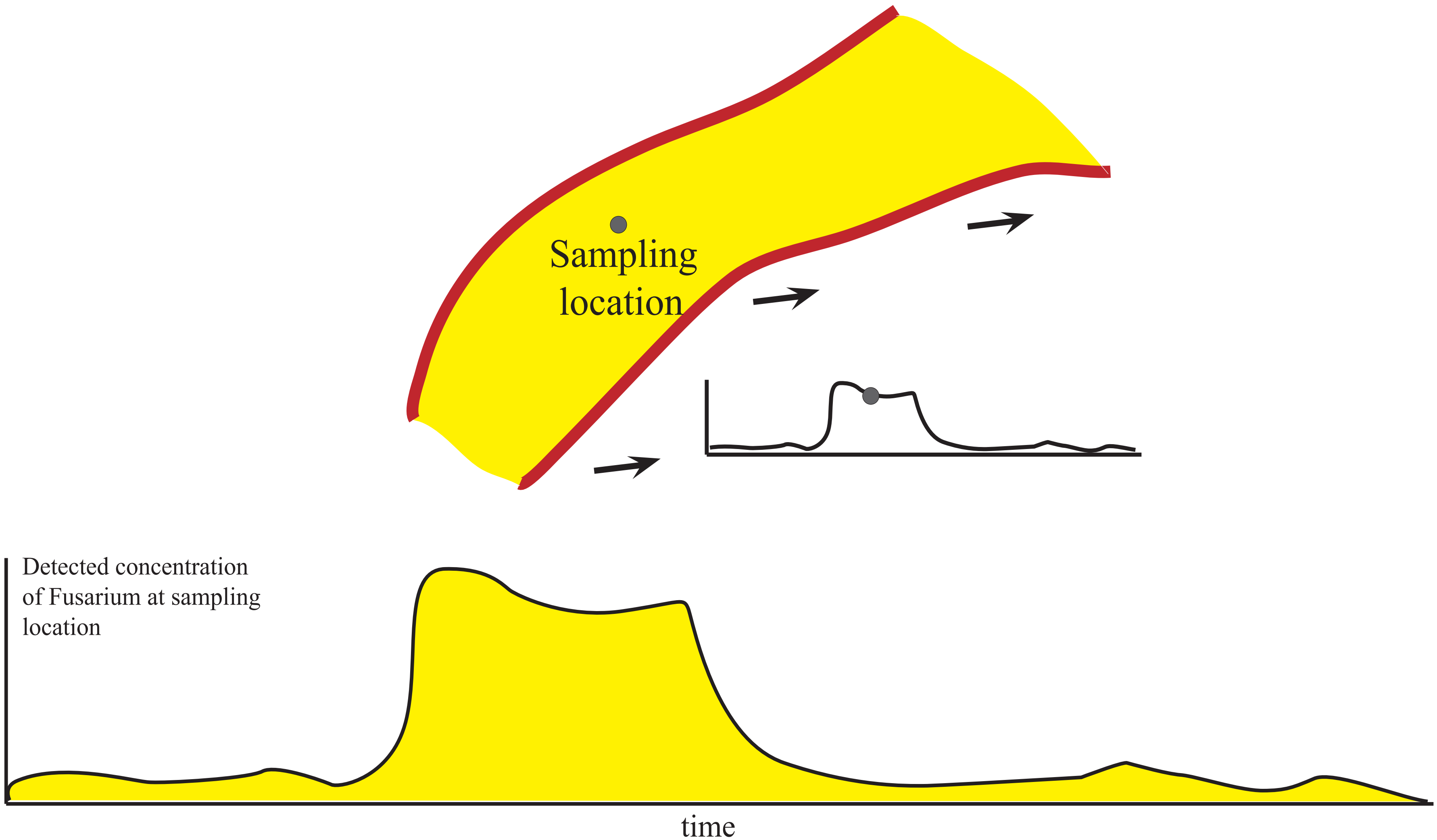

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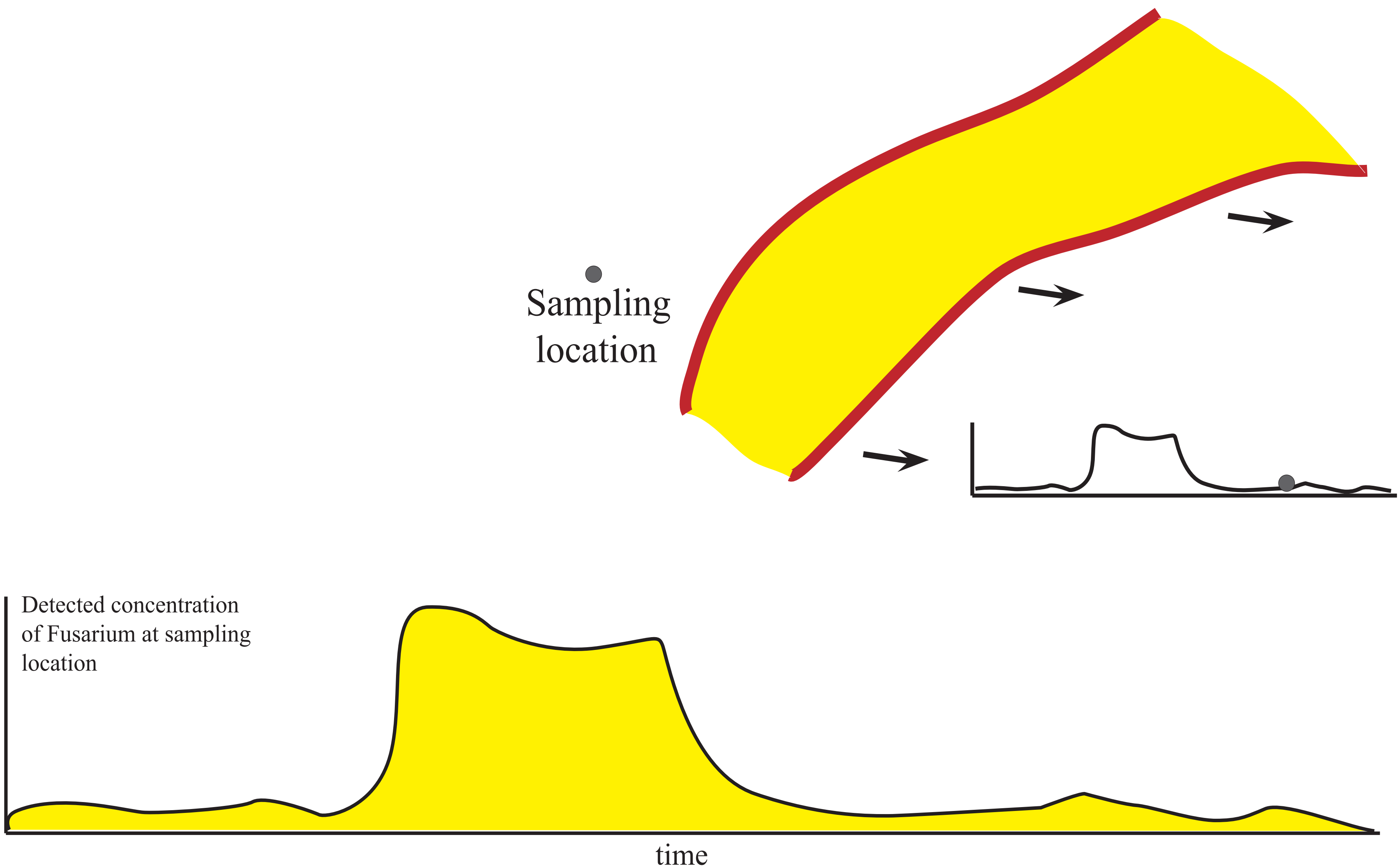
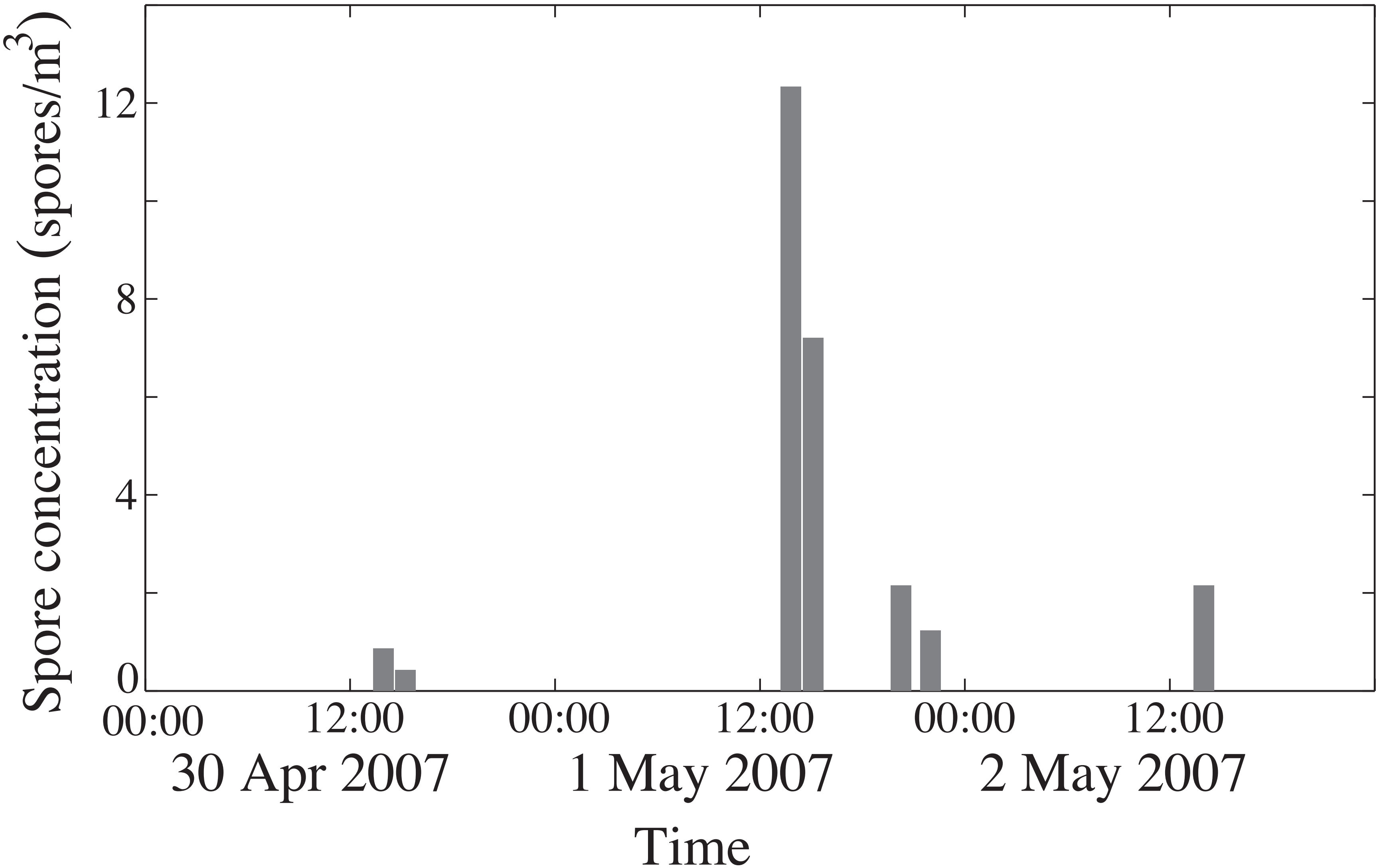
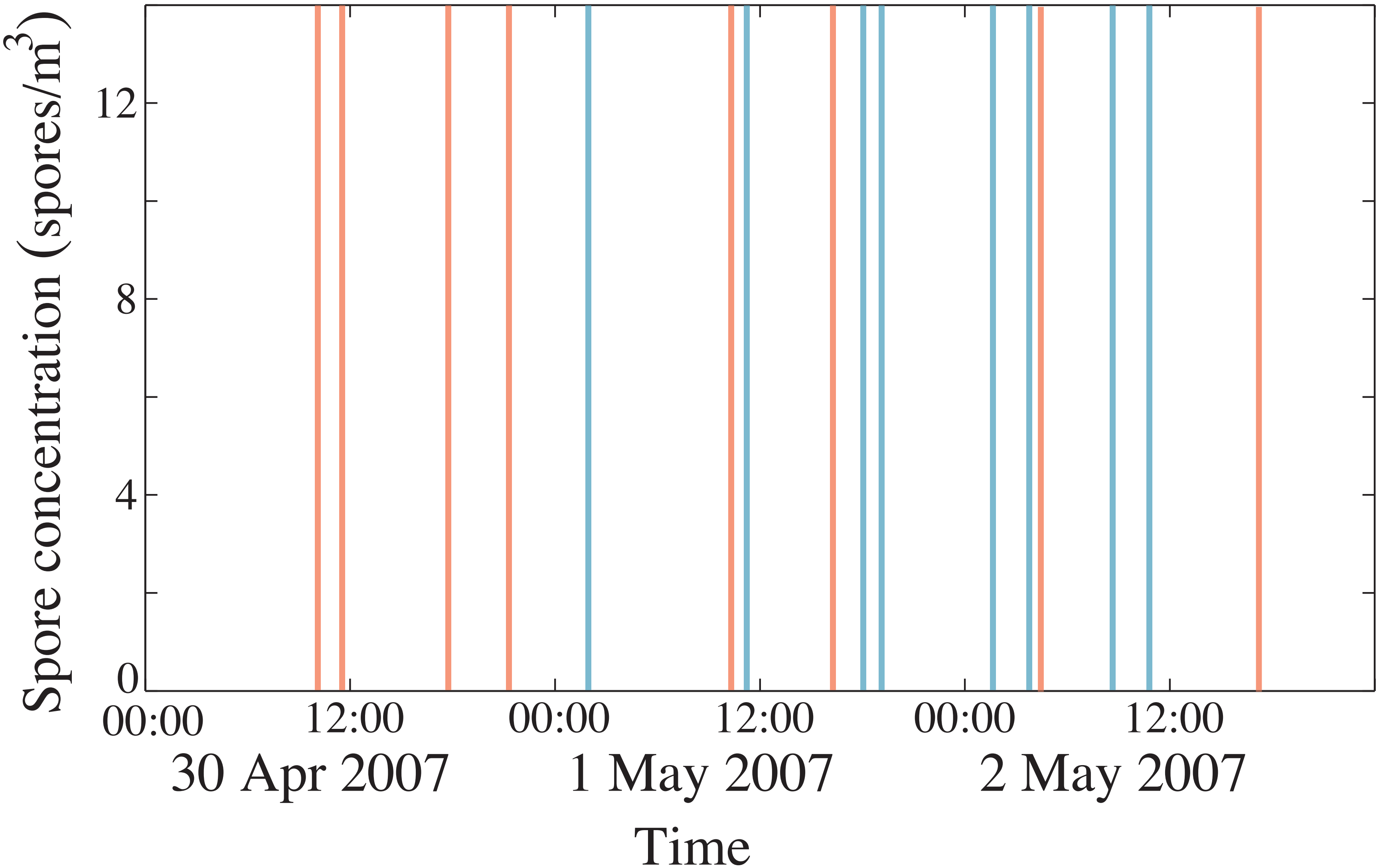
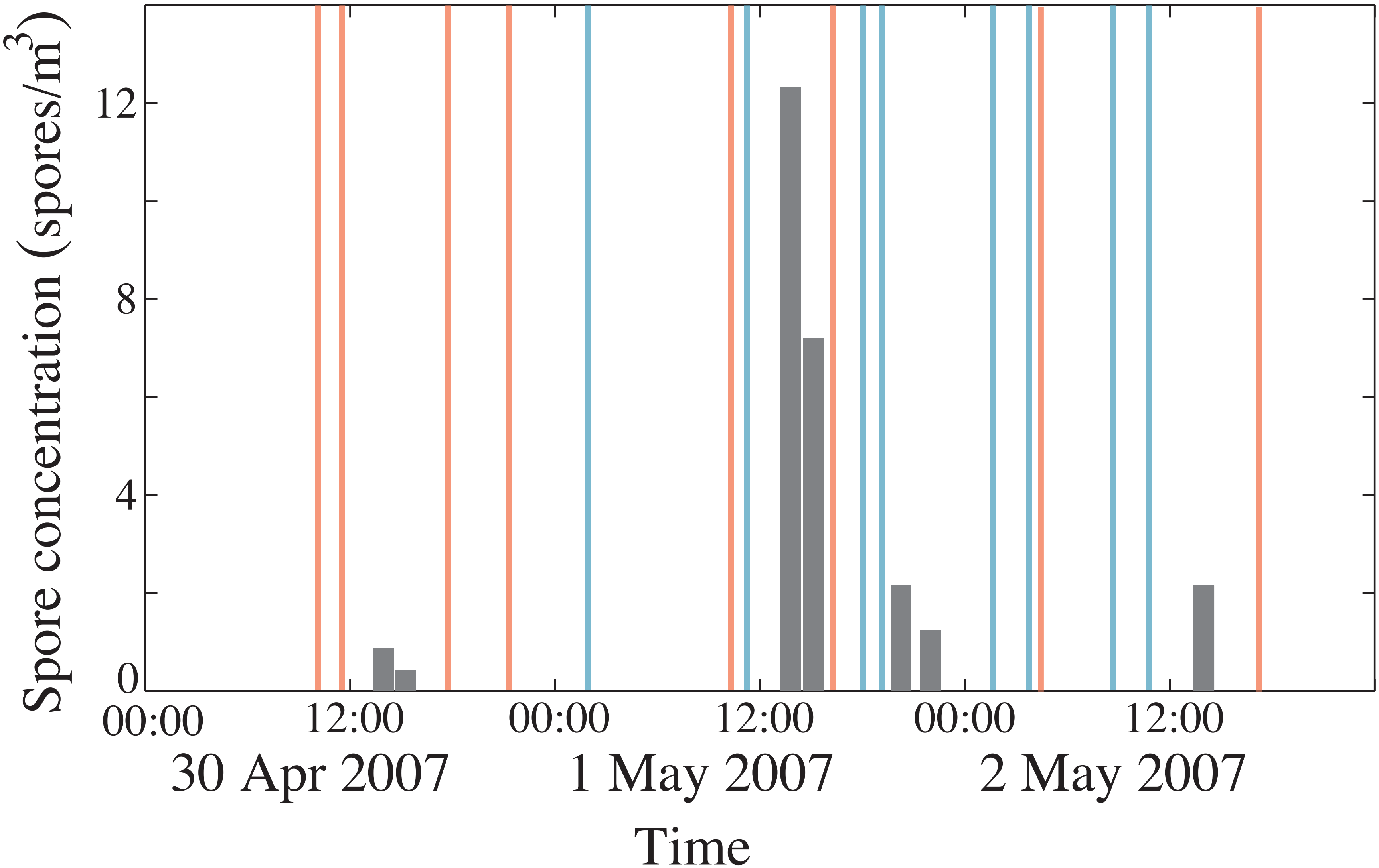




Time series of concentration $\{(t_0, C_0), \ldots, (t_{N-1}, C_{N-1})\}$



LCS passage times: orange = repelling LCSs, blue = attracting

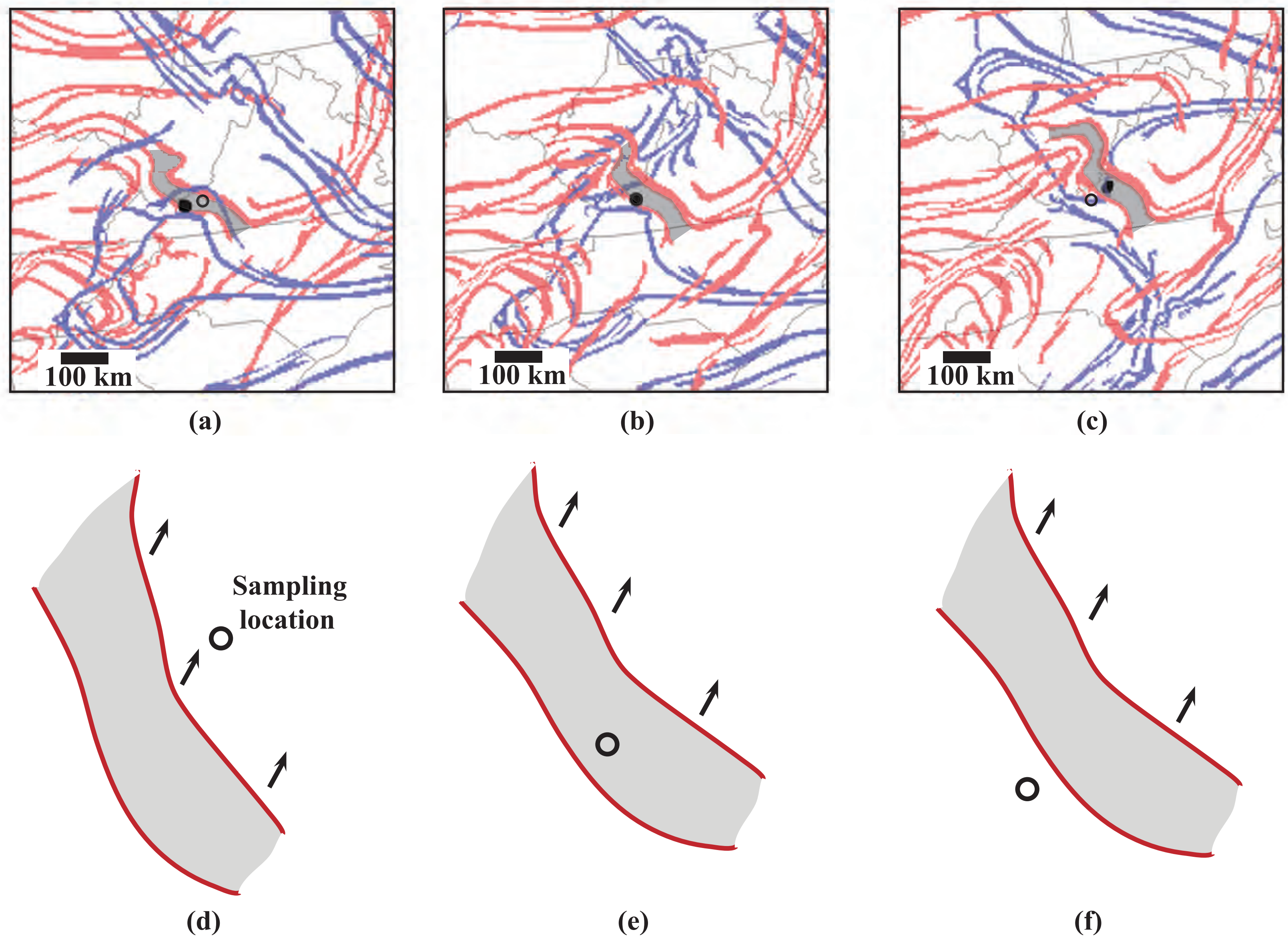


Compare: are there patterns?

Summary of Hypothesis Testing

- Of 100 samples, only 73 sample pairs within 24 hours
- Of those, 16 show punctuated changes in the concentration of *Fusarium*
- Punctuated change $\Rightarrow$ repelling LCS passage **70% of the time** $(p = 0.0017)$
- **Punctuated changes were significantly associated with the movement of a repelling LCS**
- Correlation poor for attracting LCS: punctuated change $\Rightarrow$ attracting LCS passage 37% of the time $(p = 0.33)$

Example: Filament bounded by repelling LCS



12:00 UTC 1 May 2007 15:00 UTC 1 May 2007 18:00 UTC 1 May 2007

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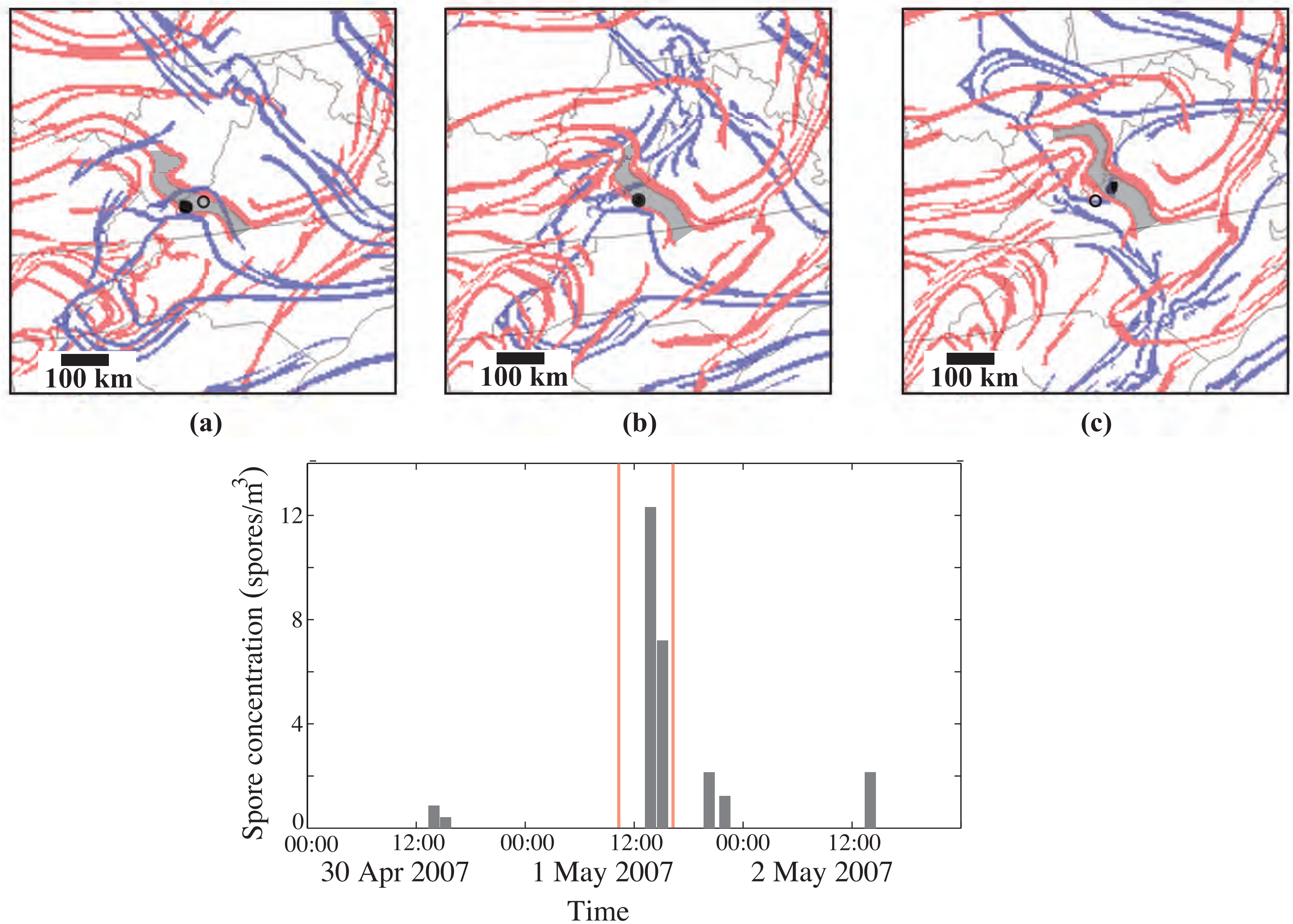


12:00 UTC 1 May 2007 15:00 UTC 1 May 2007 18:00 UTC 1 May 2007

Direct computation of coherent sets

- Take probabilistic point of view
- Partition phase space into **loosely coupled regions**

Coherent sets $\approx$ "Leaky" regions with a long residence time[3]

phase space is divided into several invariant and almost-invariant sets.

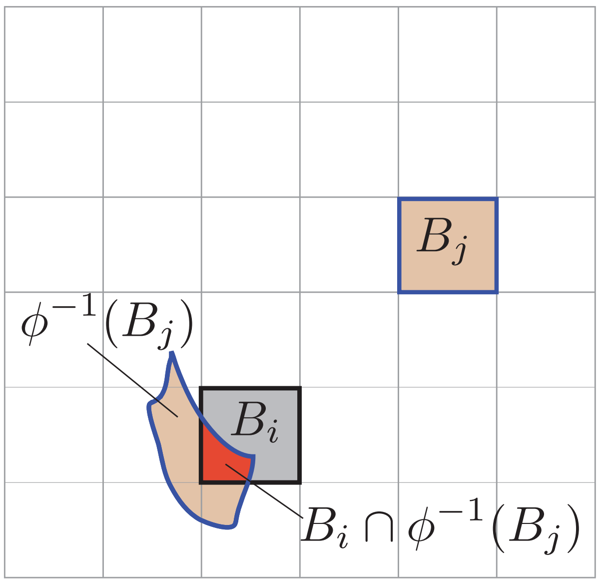
[3] work of Dellnitz, Junge, Deuflhard, Froyland, Schütte, et al

Direct computation of coherent sets

- Create box partition of phase space $\mathcal{B} = \{B_1, \dots B_q\}$, with q large
- Consider a q-by-q **transition (Ulam) matrix**, P, for our dynamical system, where

$$P_{ij} = \frac{m(B_i \cap \phi^{-1}(B_j))}{m(B_i)},$$

the *transition probability* from B_i to B_j using $\phi = \phi_t^{t+T}$

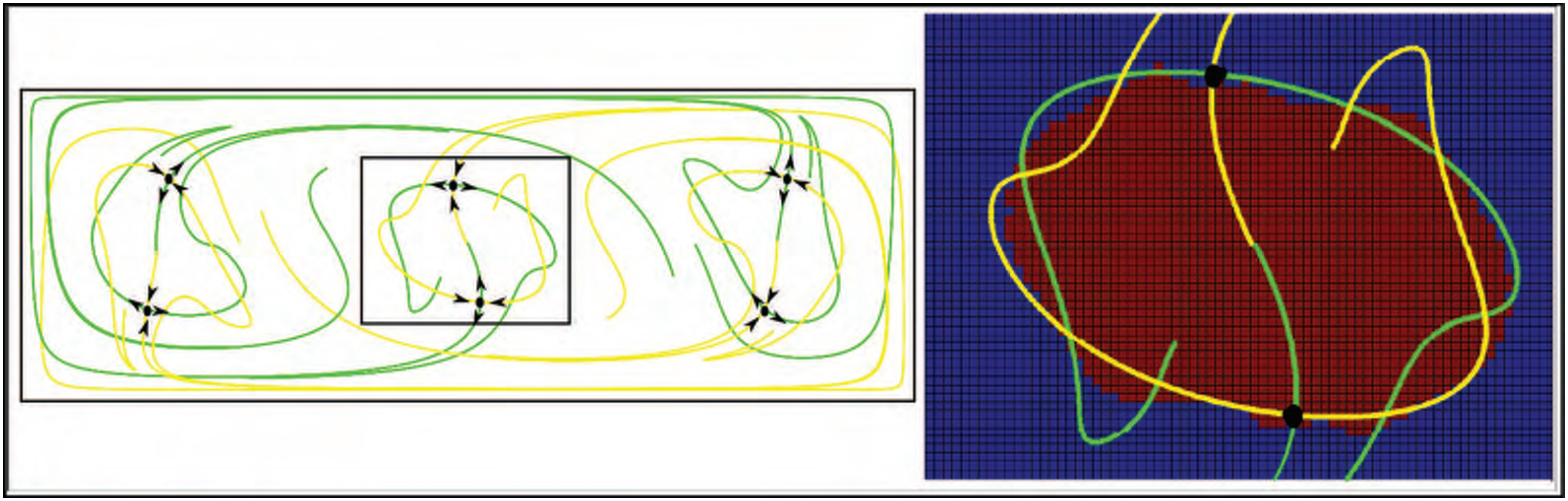


- P approximates the flow map ϕ_t^{t+T} via a finite state Markov chain.

Direct computation of coherent sets

- A set B is called *almost-invariant* over the interval $[t, t+T]$ if
$$\rho(B) = \frac{m(B \cap \phi^{-1}(B))}{m(B)} \approx 1.$$
- Can maximize value of ρ over all possible combinations of sets $B \in \mathcal{B}$.
- In practice, AISs or relatedly, almost-cyclic sets (ACSs), identified via **eigenvectors** (of eigenvalues with $|\lambda| \approx 1$) of P or graph-partitioning

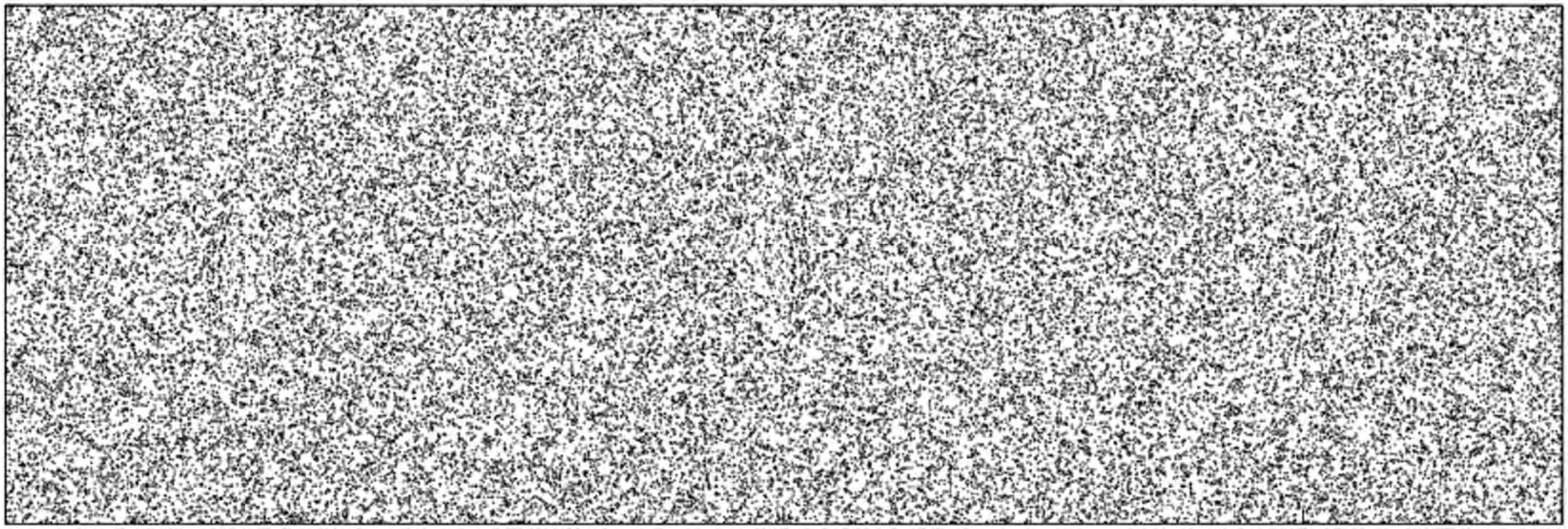
Identifying coherent sets



- Consider lid-driven cavity flow system with system parameter τ_f
- For $\tau_f > 1$, periodic points and manifolds exist
- Coherent set boundaries are manifolds of periodic points
- Known previously[4] and applies to more general objects than periodic points, i.e. normally hyperbolic invariant manifolds (NHIMs)

[4] Dellnitz, Junge, Lo, Marsden, Padberg, Preis, Ross, Thiere [2005] Phys. Rev. Lett.; Dellnitz, Junge, Koon, Lekien, Lo, Marsden, Padberg, Preis, Ross, Thiere [2005] Int. J. Bif. Chaos

Identifying coherent sets



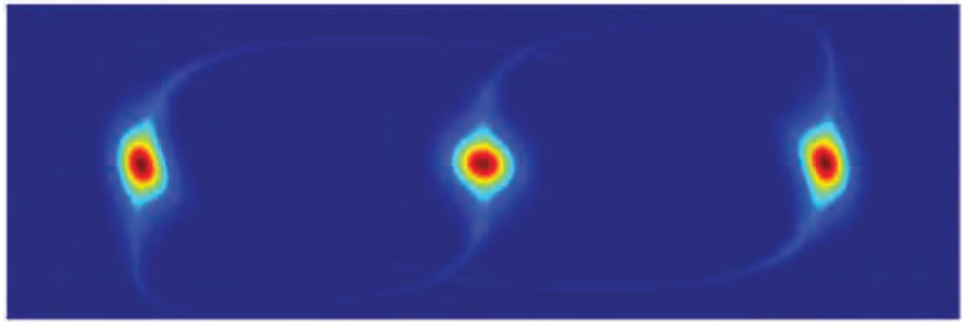
Poincaré section for $\tau_f < 1 \Rightarrow$ no obvious structure!

- For $\tau_f < 1$, no periodic orbits of low period known
- Is the phase space featureless?

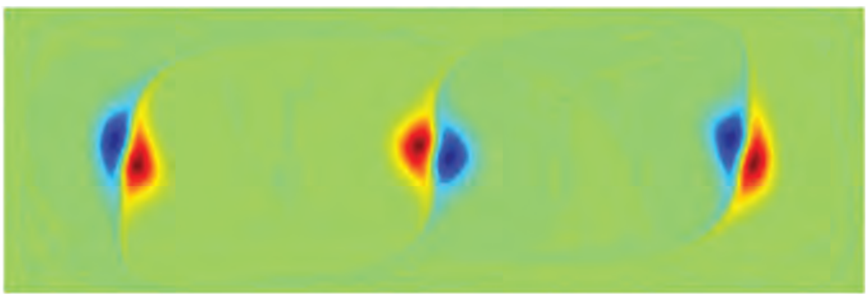
- Consider transition matrix $P_t^{t+\tau_f}$ induced by Poincaré map $\phi_t^{t+\tau_f}$

Identifying coherent sets

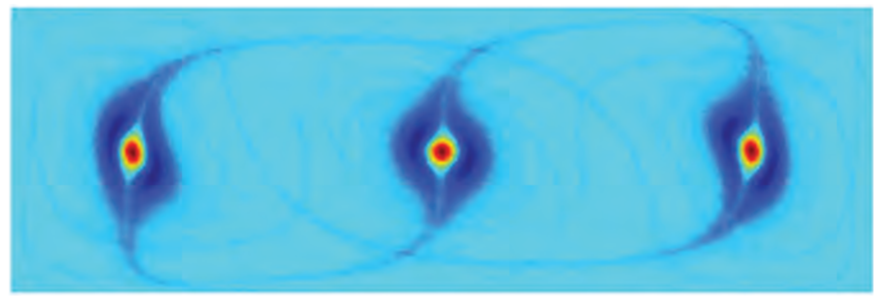
Top eigenvectors for $\tau_f = 0.99$ reveal hierarchy of phase space structures



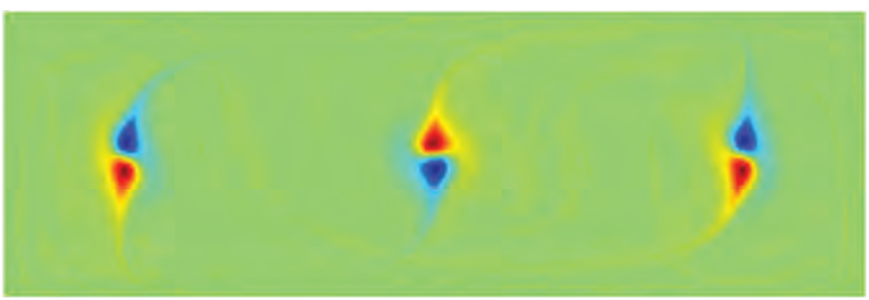
ν_2



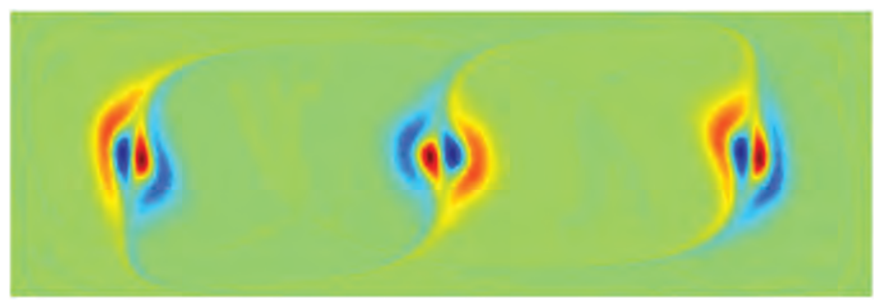
ν_3



ν_4

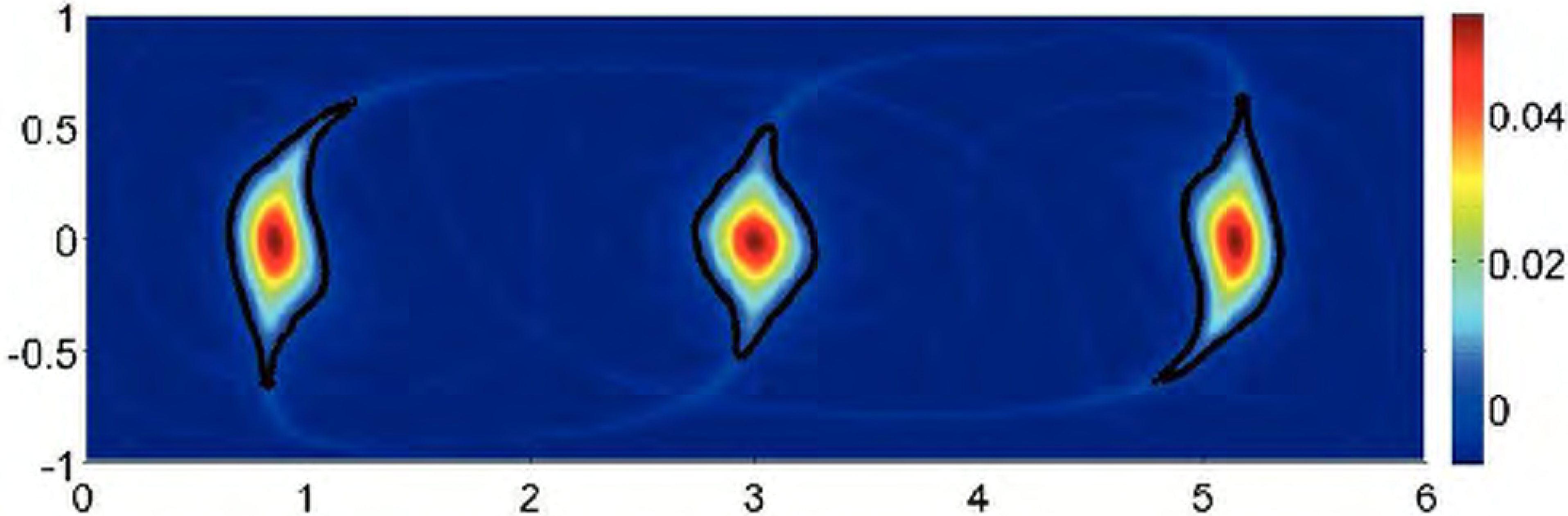


ν_5



ν_6

Identifying coherent sets



The zero contour (black) is the boundary between the two almost-invariant sets.

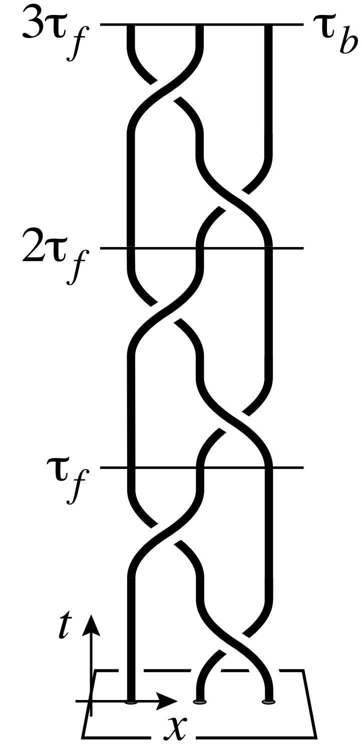
- Three almost-cyclic coherent sets (ACSs) of period 3
- ACSs effectively replace compact region bounded by saddle manifolds
- Also: we see a **dynamical remnant of the global 'stable and unstable manifolds' of the saddle points**, even there are no more saddle points

Identifying coherent sets

Almost-cyclic sets stirring the surrounding fluid like ‘ghost rods’
— **works even when periodic orbits are absent!**

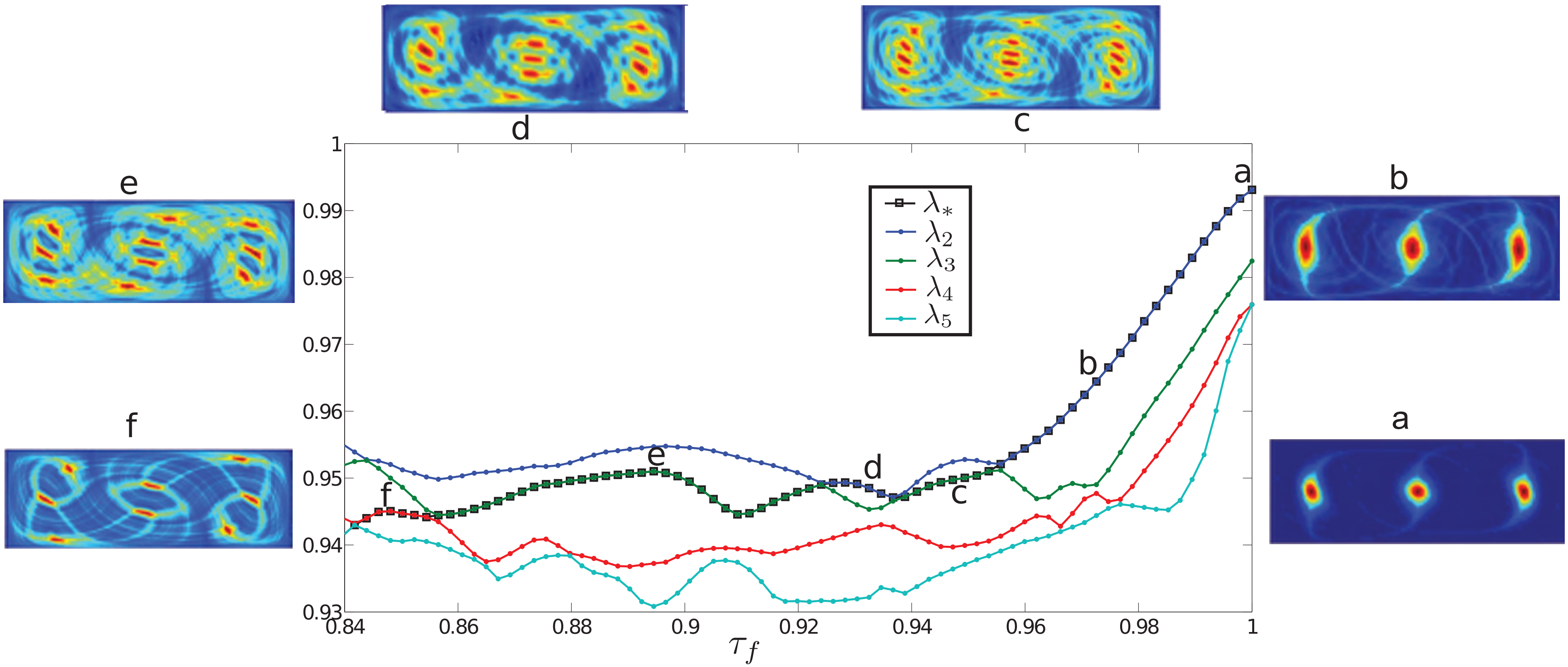
Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

Identifying coherent sets



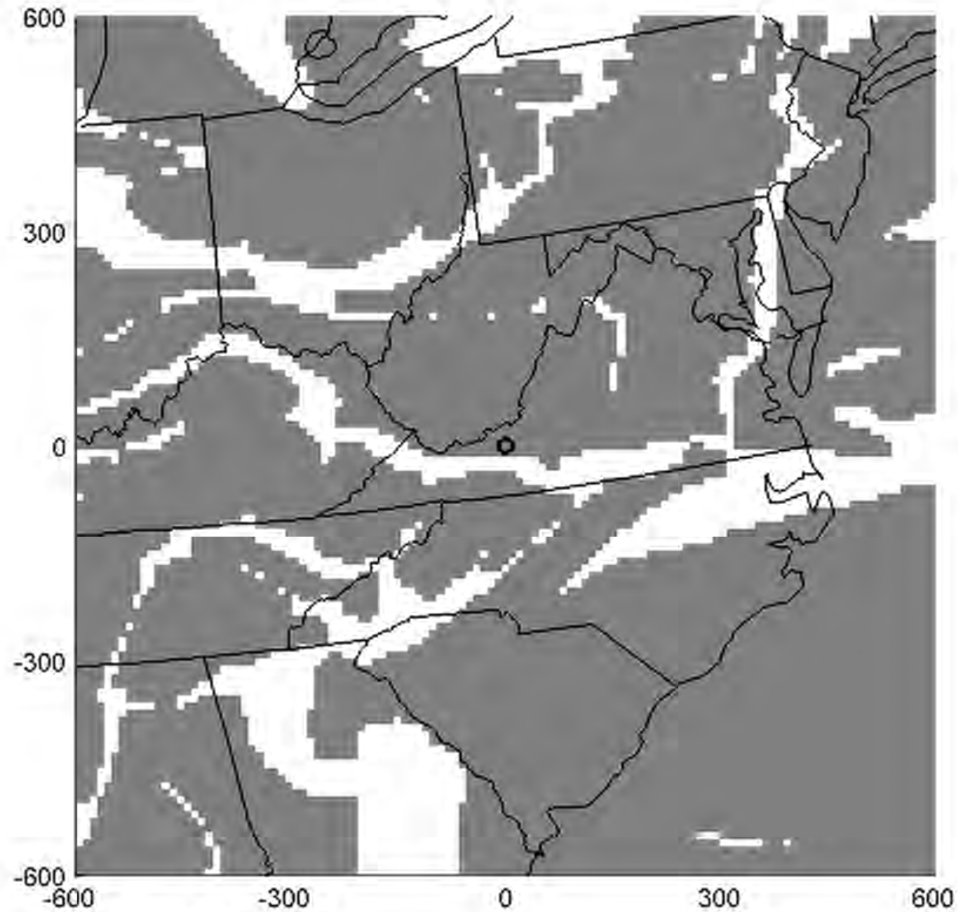
Braid of ACSs gives lower bound of entropy via Thurston-Nielsen
— One only needs approximately cyclic blobs of fluid
— Even though the theorems require exactly periodic points!
— Stremler, Ross, Grover, Kumar [2011] Phys. Rev. Lett.

Eigenvalues/eigenvectors vs. bifurcation parameter



Movie shows change in eigenvector along branch marked with '$-\square-$' above (from a to f), as τ_f decreases $\Rightarrow$

Coherent sets in the atmosphere



- Coherent sets during 24 hours starting 09:00 1 May 2007

Final words on coherent sets & transport

- What are the robust descriptions of coherent sets and transport which work in aperiodic, finite-time settings?

- Methods for finding boundaries of coherent sets and coherent sets themselves are fruitful for illuminating structure and transport.

- Lobe dynamics, finite-time symbolic dynamics...

- In analogy with point vortices, can we find equations of motion for generalized coherent sets and their influence on each other?

The End

For papers, movies, etc., visit:
www.shaneross.com

Main Papers:

- Ross & Tallapragada [2011] Detecting and exploiting chaotic transport in mechanical systems, *Applications of Chaos and Non-linear Dynamics in Engineering - Vol. 2*, Springer.
- Tallapragada & Ross [2011] A geometric and probabilistic description of coherent sets, *under review*.
- Tallapragada, Ross, Schmale [2011] Lagrangian coherent structures are associated with fluctuations in airborne microbial populations. *Chaos* 21, 033122.
- Stremler, Ross, Grover, Kumar [2011] Topological chaos and periodic braiding of almost-cyclic sets. *Physical Review Letters* 106, 114101.
- Senatore & Ross [2011] Detection and characterization of transport barriers in complex flows via ridge extraction of the finite time Lyapunov exponent field, *International Journal for Numerical Methods in Engineering* 86, 1163.
- Lekien & Ross [2010] The computation of finite-time Lyapunov exponents on unstructured meshes and for non-Euclidean manifolds. *Chaos* 20, 017505.