Geometric and probabilistic descriptions of chaotic phase space transport: stirring by braiding of almost-cyclic sets

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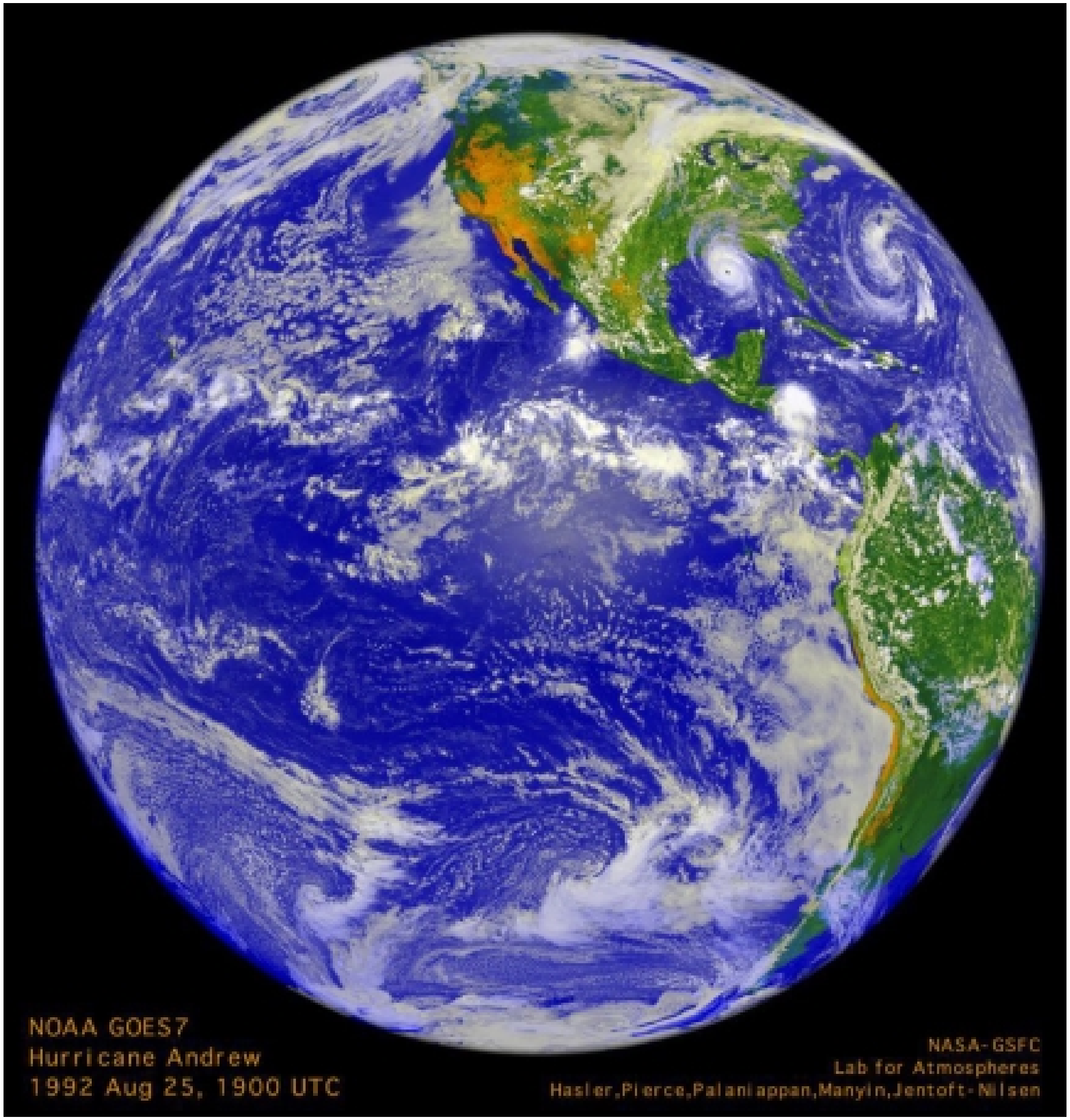
BIRS Workshop, Open Dynamical Systems, April 2012

Motivation: complex fluid motion, mixing, and control



Motivation: complex fluid motion, mixing, and control

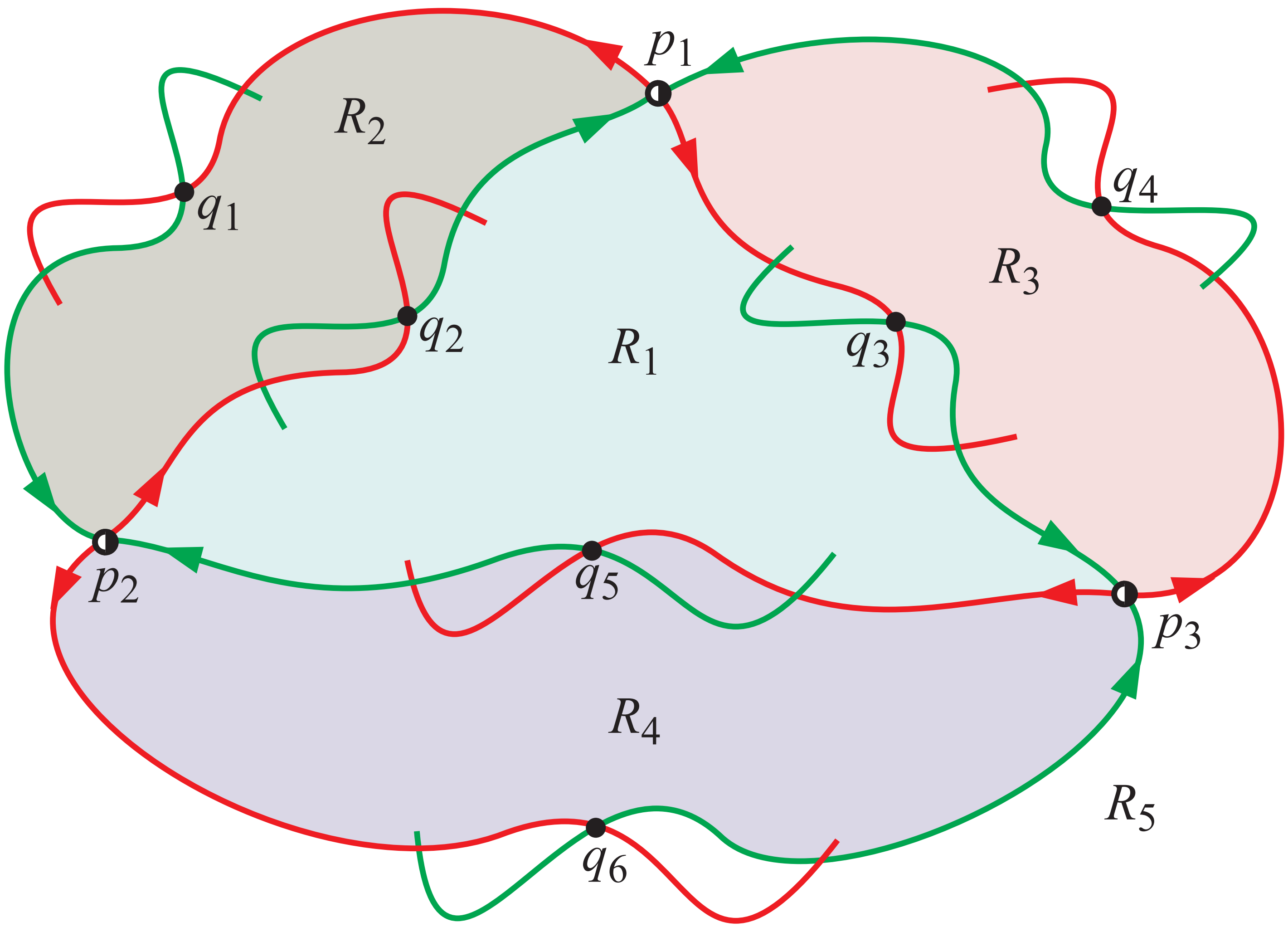
Atmosphere over North America. Lagrangian coherent boundaries: orange = repelling, blue = attracting

Motivation: complex fluid motion, mixing, and control

Table top fluid experiment. Lagrangian coherent boundaries: red = repelling, blue = attracting

Motivation: complex fluid motion, mixing, and control

- Selectively 'jumping' between coherent sets using control
- Moving between mobile subregions of different finite-time itineraries



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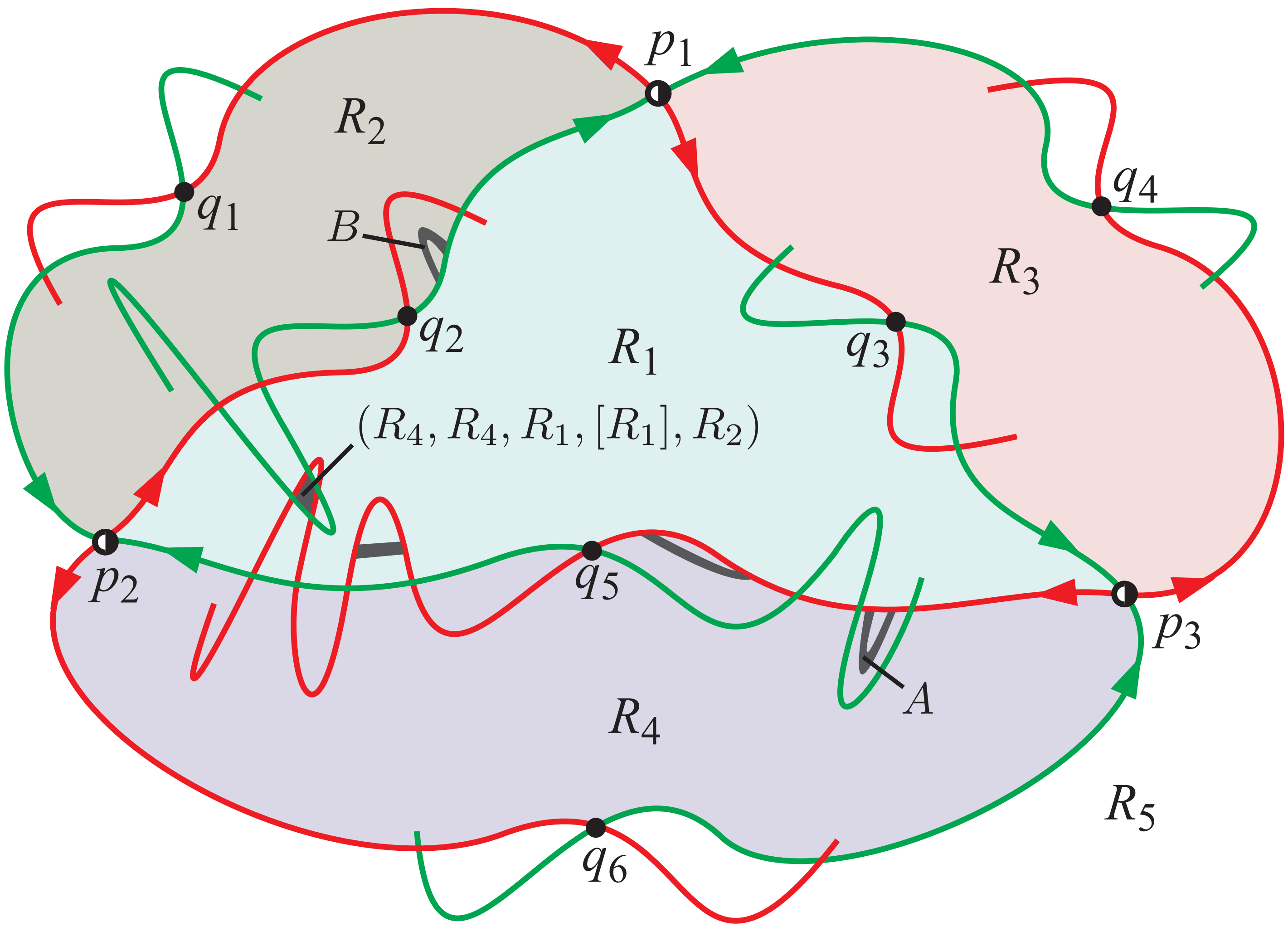

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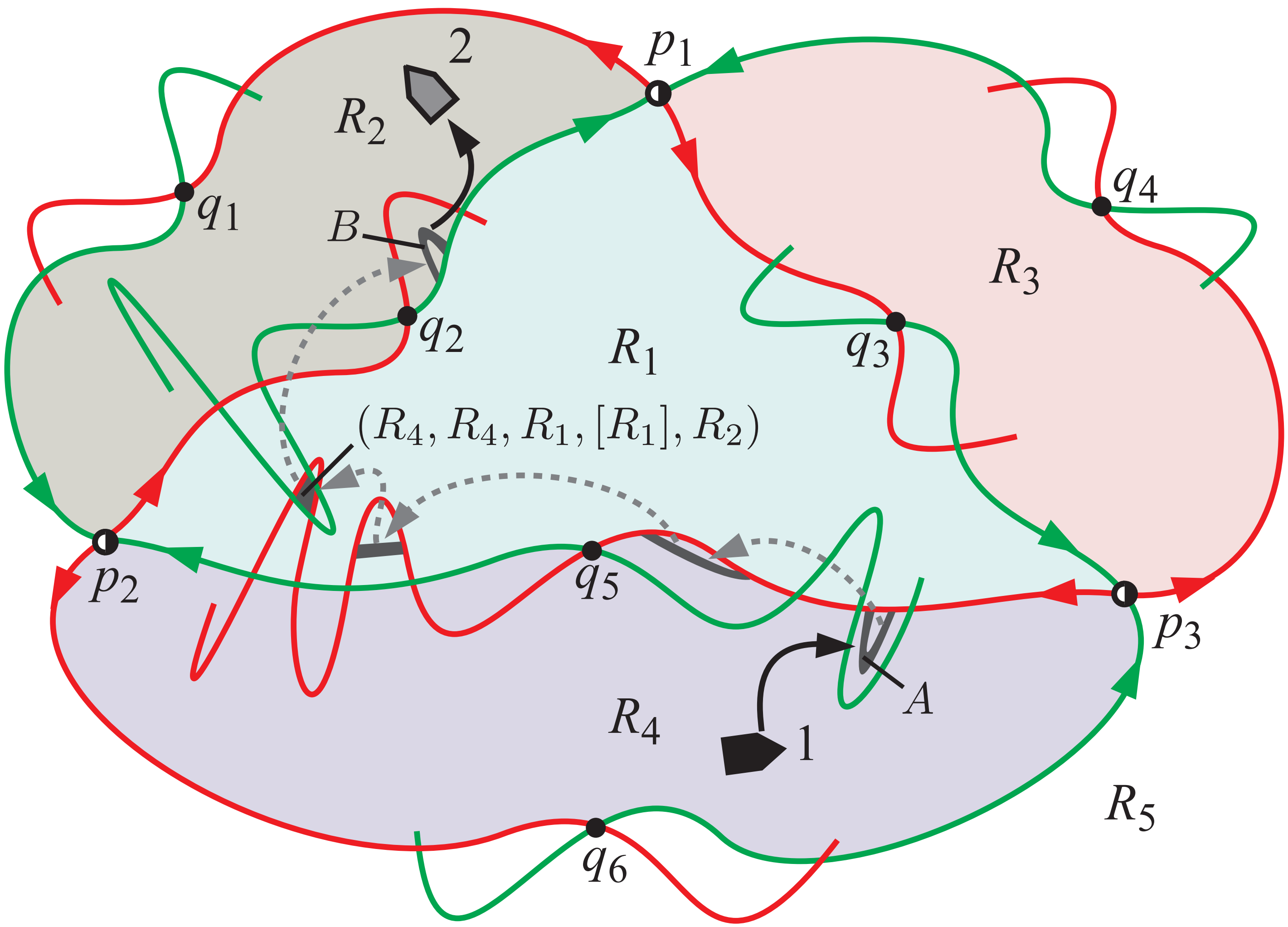


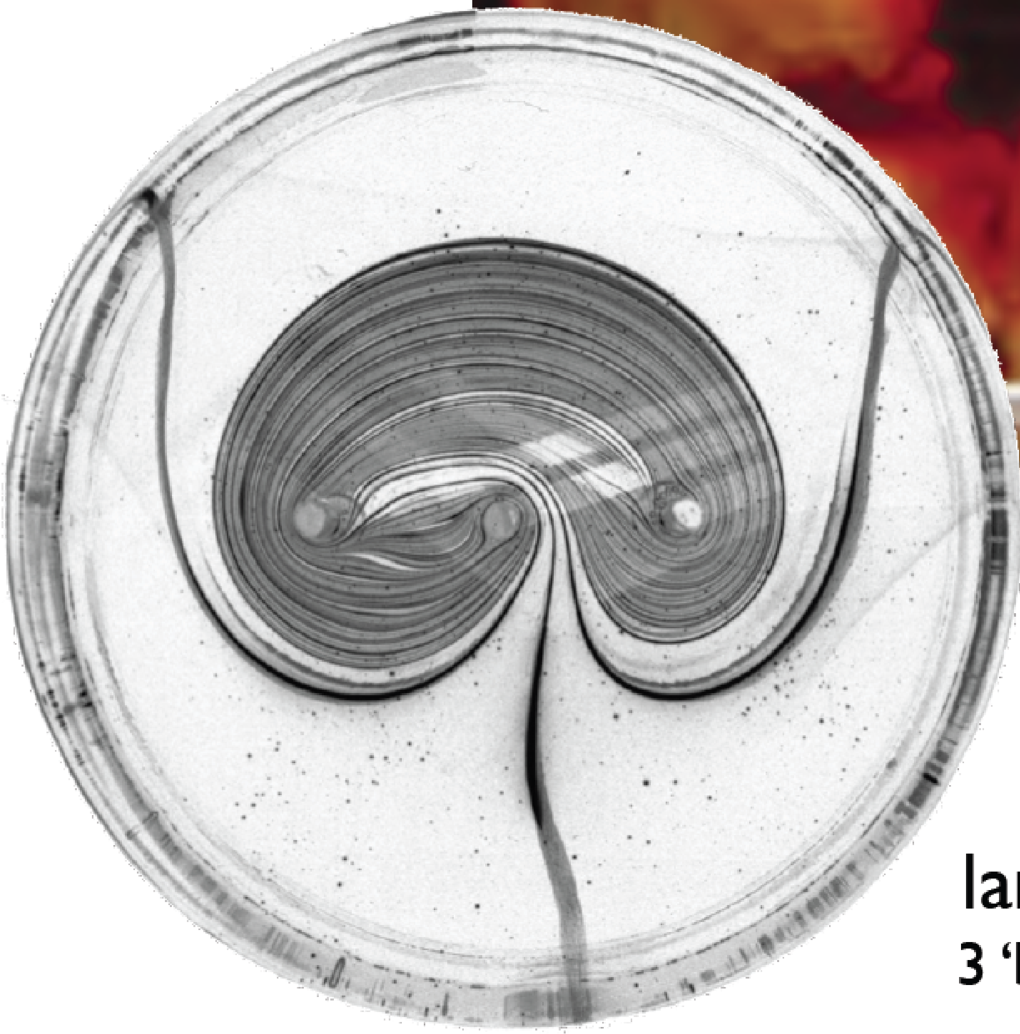
Motivation: complex fluid motion, mixing, and control

- Selectively 'jumping' between coherent sets using control
- Moving between mobile subregions of different finite-time itineraries

green=uncontrolled, red=controlled

Stirring fluids with solid rods

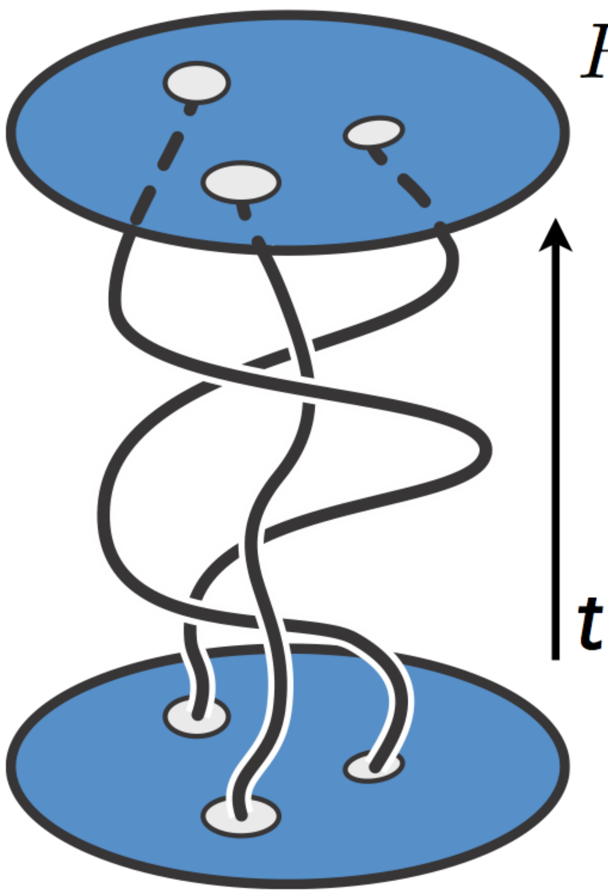
turbulent mixing
spoon in coffee



laminar mixing
3 'braiding' rods in glycerin

Topological chaos through braiding of stirrers

- Topological chaos is 'built in' the flow due to the topology of boundary motions

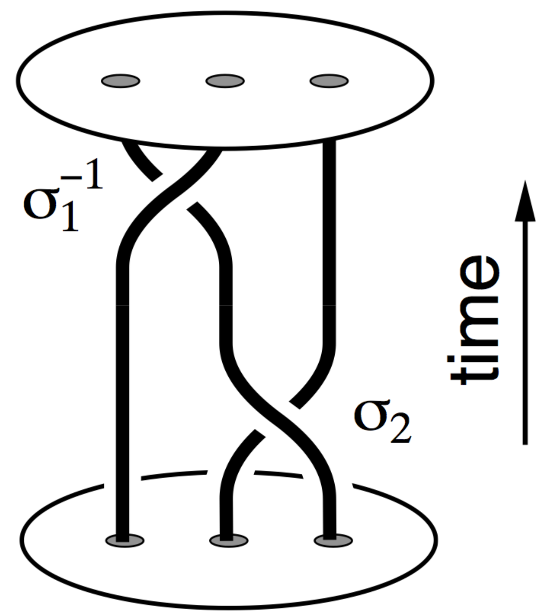


R_N : 2D fluid region with N stirring 'rods'

- stirrers move on periodic orbits
- stirrers = solid objects or *fluid particles*
- stirrer motions generate diffeomorphism
  $f : R_N \to R_N$
- stirrer trajectories generate braids in 2+1 dimensional space-time

Thurston-Nielsen classification theorem

- Thurston (1988) Bull. Am. Math. Soc.
- A stirrer motion f is isotopic to a stirrer motion g of one of three types (i) finite order (f.o.): the nth iterate of g is the identity (ii) pseudo-Anosov (pA): g has Markov partition with transition matrix A, topological entropy $h_{\mathrm{TN}}(g) = \log(\lambda_{PF}(A))$, where $\lambda_{\mathrm{PF}}(A) > 1$ (iii) reducible: g contains both f.o. and pA regions



- h_{TN} computed from 'braid word', e.g., $\sigma_1^{-1}\sigma_2$
- $\log(\lambda_{PF}(A))$ provides a **lower bound** on the true topological entropy

Topological chaos in a viscous fluid experiment

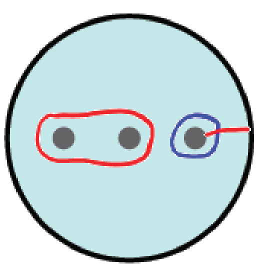
Move 3 rods on 'figure-8' paths through glycerin

Boyland, Aref & Stremler (2000) *J. Fluid Mech.*

- stirrers move on periodic orbits in two steps
- Thurston-Nielsen theorem gives a lower bound on stretching:

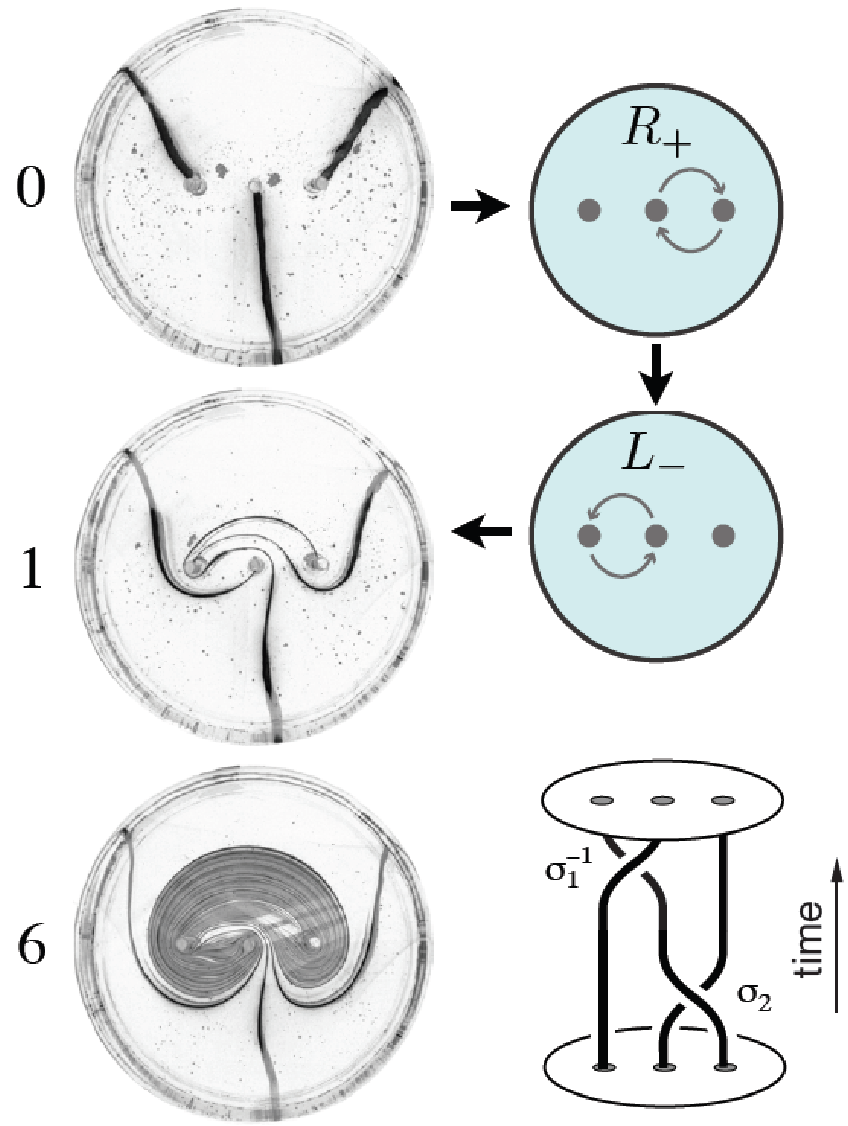
$$\lambda_{\mathrm{TN}} = \tfrac{1}{2}\left(3 + \sqrt{5}\right)$$

$$h_{\mathrm{TN}} = \log(\lambda_{\mathrm{TN}}) = 0.962\ldots$$

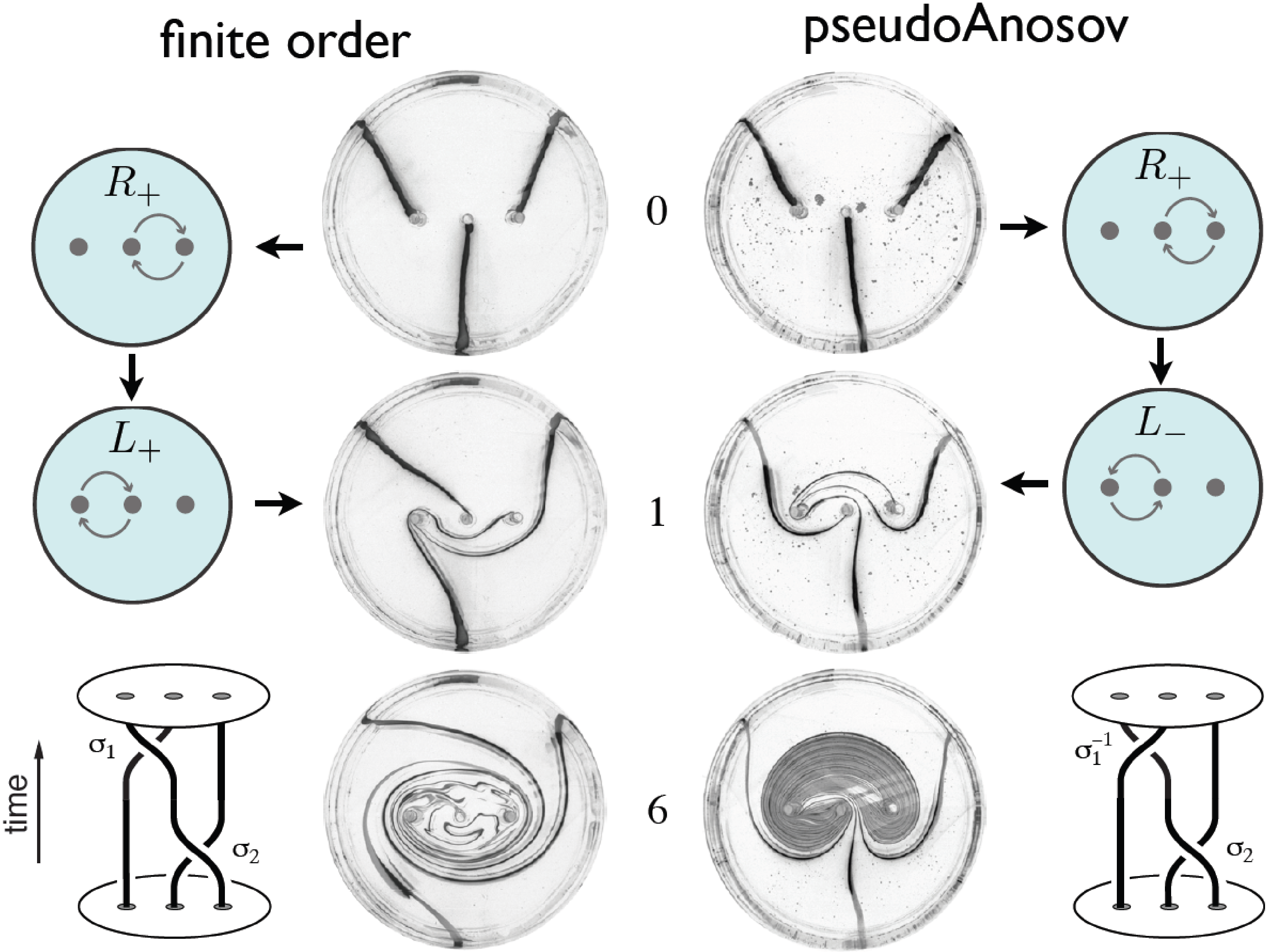


non-trivial material lines grow like $l \sim l_0\ \lambda^n$

$$\lambda \geq \lambda_{\mathrm{TN}}$$



Topological chaos in a viscous fluid experiment



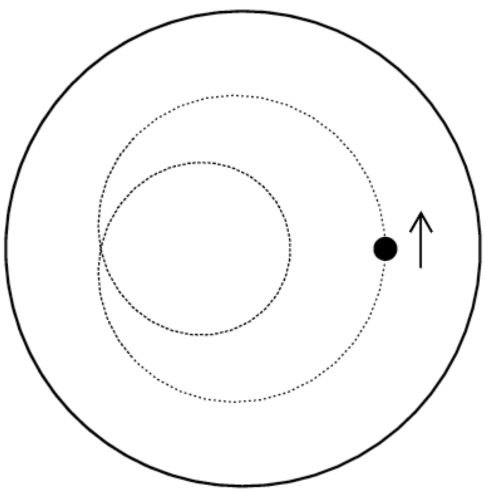
‘Stirring’ with fluid particles

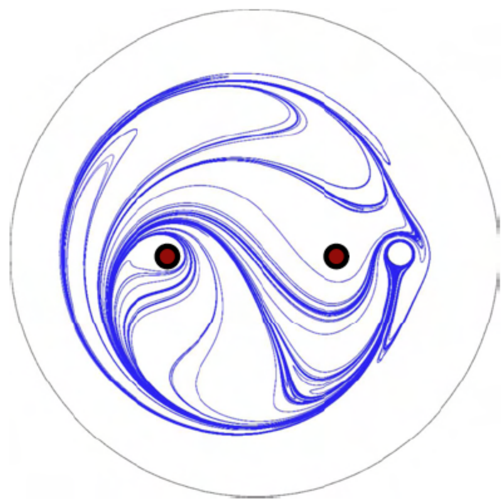
point vortices in a periodic domain

Boyland, Stremler & Aref (2003) *Physica* D

one rod moving on an epicyclic trajectory

Gouillart, Thiffeault & Finn (2006) *Phys. Rev.* E





‘ghost rods’

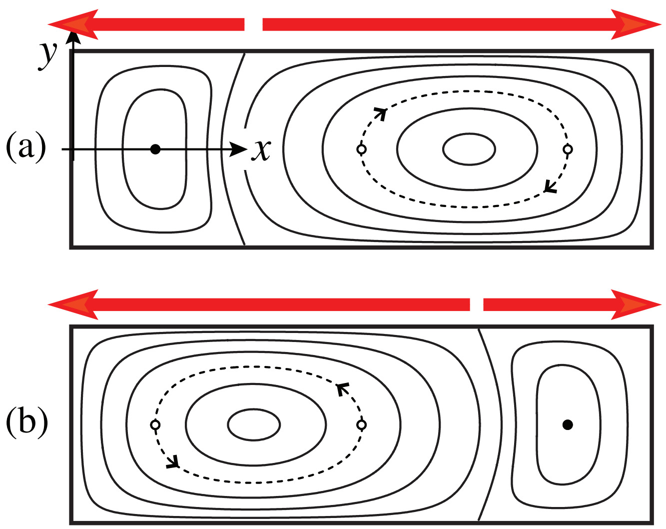
solid rods

Fluid is wrapped around ‘ghost rods’ in the fluid

– *flow structure assists in the stirring*

Ghost rods in microfluidics mixer

- Lid-driven cavity flow, periodic vector field



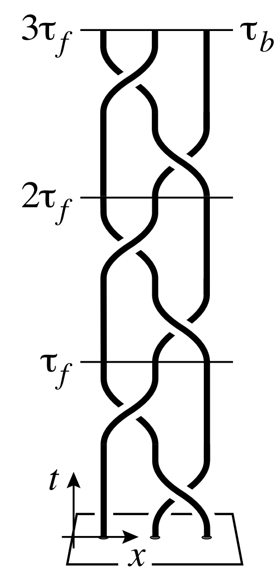
streamlines for $\tau_f = 1$

tracer blob $(\tau_f > 1)$

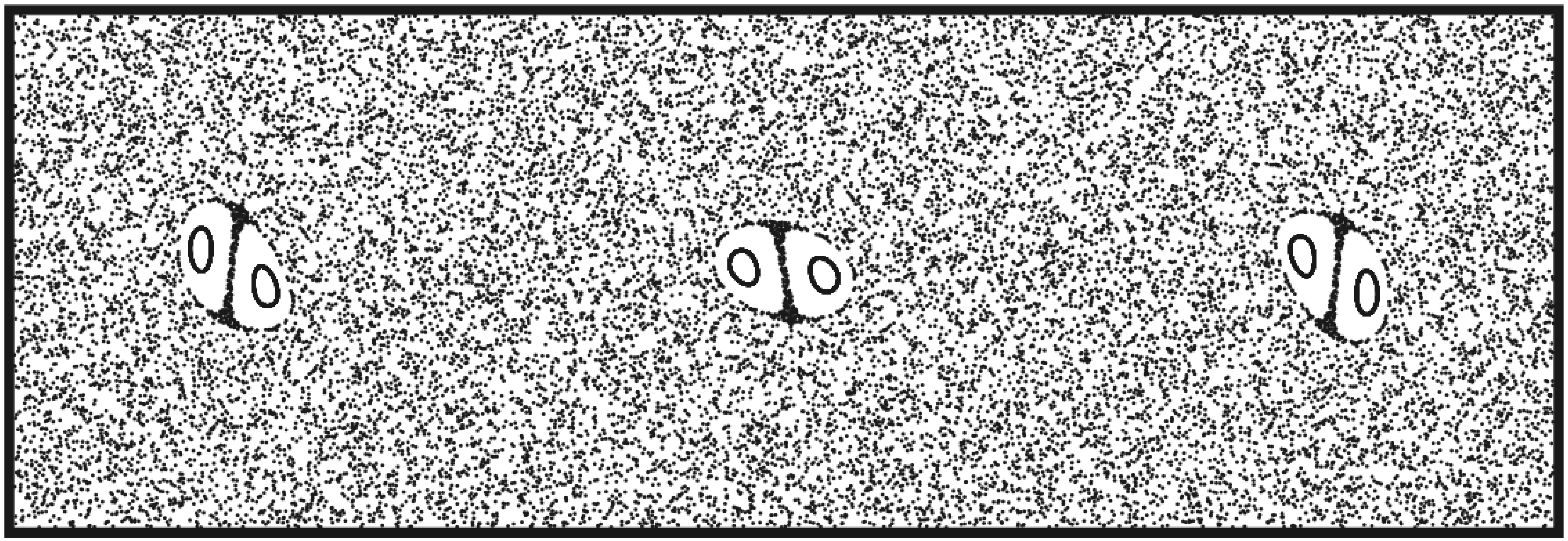
- $t \in [n\tau_f, (n+1)\tau_f/2)$, right two points exchange clockwise
- $t \in [(n+1)\tau_f/2, (n+1)\tau_f)$, left two points exchange counter-clockwise
- System has parameter τ_f, which we treat as a bifurcation parameter — critical point $\tau_f^* = 1$

Stirring protocol ⇒ braid ⇒ topological entropy

- Consider period-τ_f map
- For $\tau_f = 1$, period 3 points act as 'ghost rods'
- Their braid ⇒ $h_{\mathrm{TN}} = 0.96242$ from TNCT
- Actual $h_{\mathrm{flow}} \approx 0.964$ obtained numerically
- ⇒ h_{TN} is an excellent lower bound

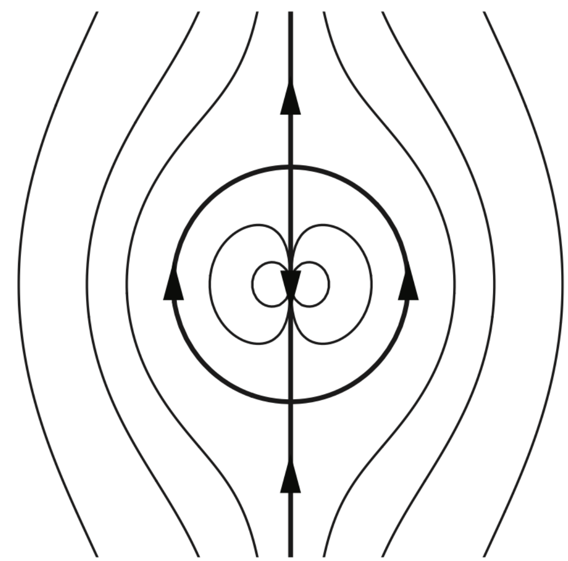


Identifying 'ghost rods': periodic points



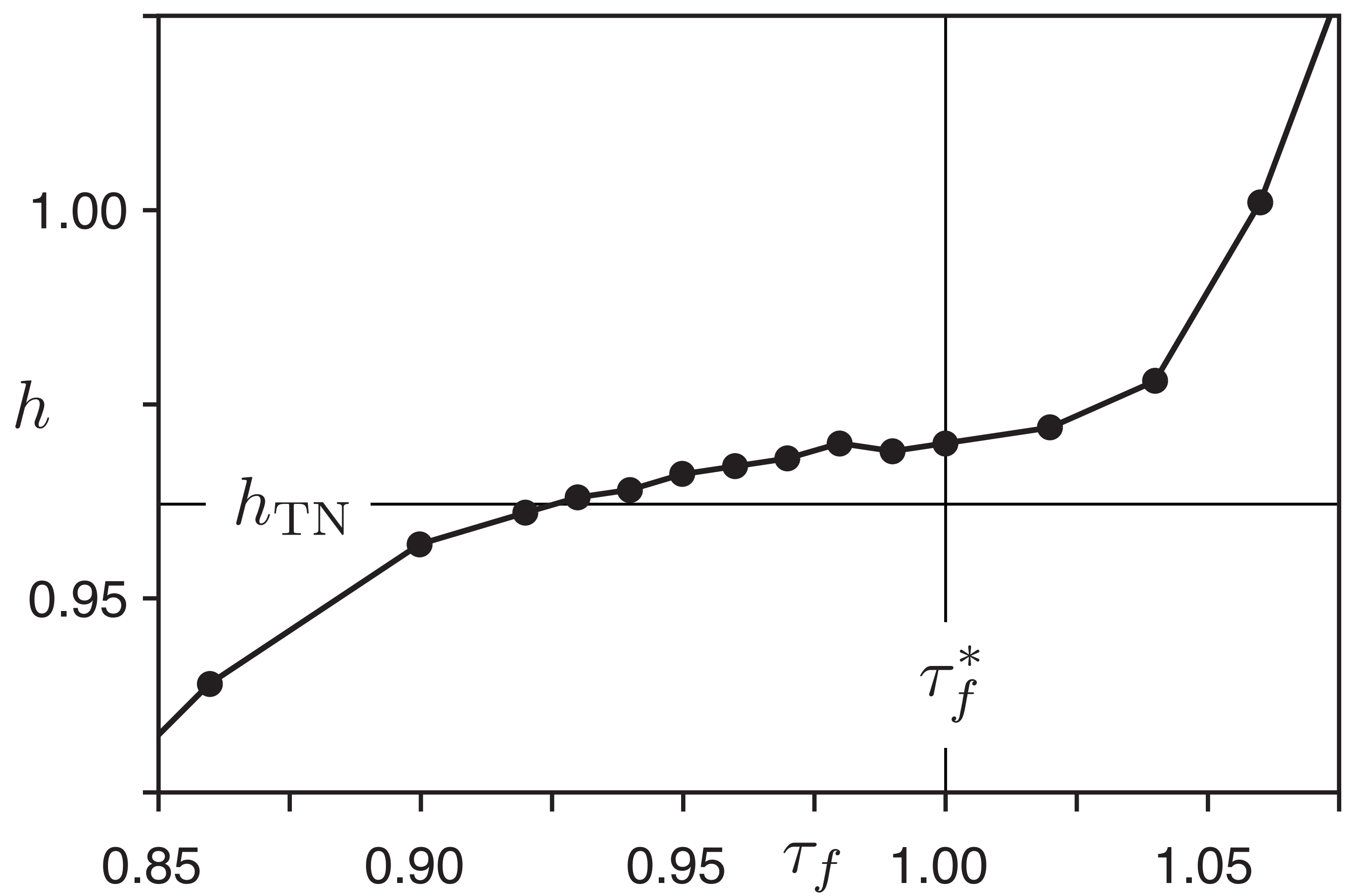
period-τ_f map for τ_f just above 1

- At $\tau_f = 1$, parabolic period 3 points of map
- $\tau_f > 1$, **elliptic / saddle points** of period 3 — streamlines around groups resemble fluid motion around a solid rod ⇒
- $\tau_f < 1$, **periodic points vanish**



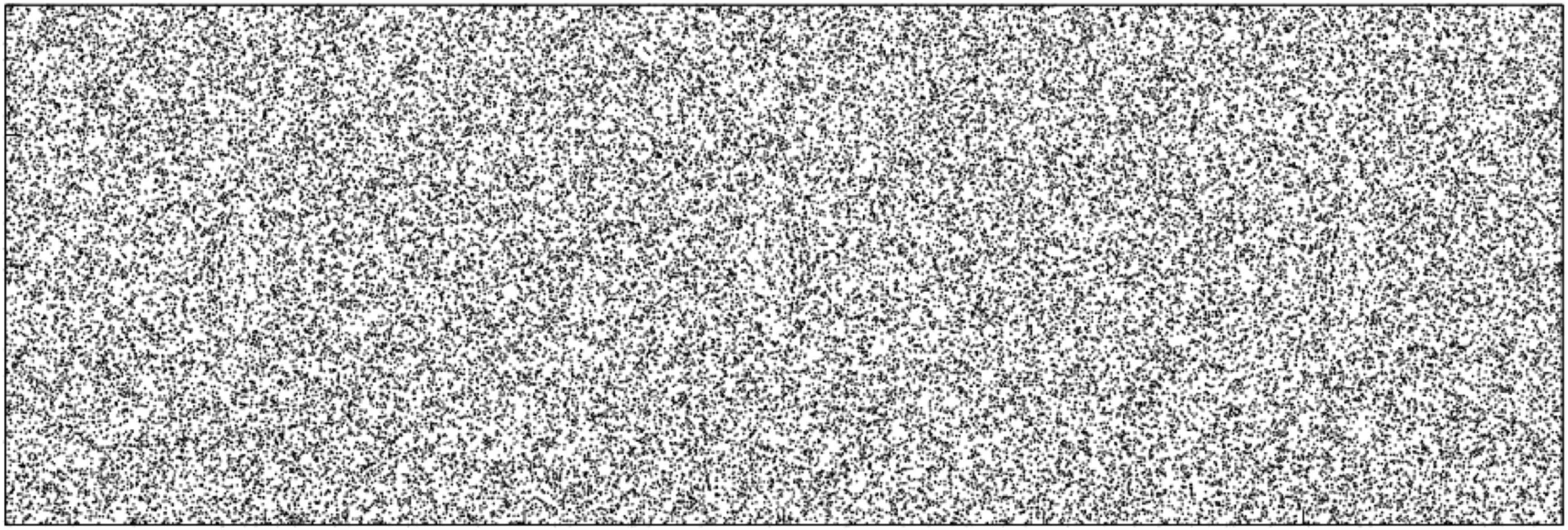
Topological entropy continuity across critical point

- Consider $\tau_f < 1$



topological entropy as a function of τ_f

Identifying 'ghost rods'?



period-τ_f map for $\tau_f < 1 \Rightarrow$ no 'obvious' structure

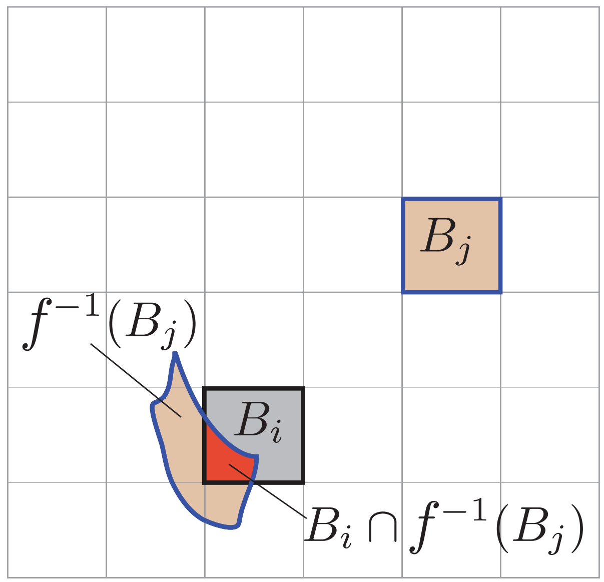
- Note the absence of any elliptical islands
- No periodic orbits of low period were found
- In practice, even when such low-order periodic orbits exist, they can be difficult to identify
- But phase space is not featureless

Almost-cyclic set approach

- Identify **almost-invariant sets** (AISs, as discussed in previous talks)
- Relatedly, **almost-cyclic sets** (ACSs) (Dellnitz & Junge [1999])
- Create box partition of phase space $\mathcal{B} = \{B_1, \ldots B_q\}$, with q large
- Consider a q-by-q **Ulam-Galerkin matrix**, P, where

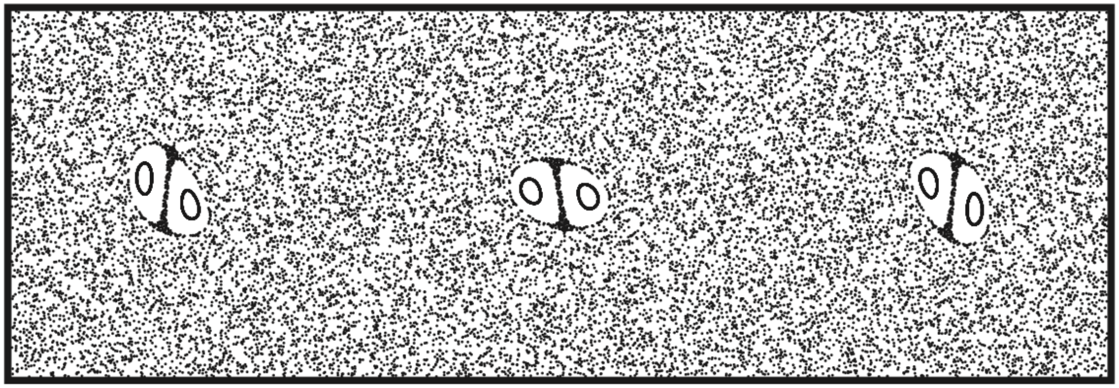
$$P_{ij} = \frac{m(B_i \cap f^{-1}(B_j))}{m(B_i)},$$

the *transition probability* from B_i to B_j using, e.g., $f = \phi_t^{t+T}$, computed numerically



- Identify AISs and ACS via spectrum of P

Identifying 'ghost rods': almost-cyclic sets



- For $\tau_f > 1$ case, where periodic points and manifolds exist...
- Agreement between ACS boundaries and manifolds of periodic points
- Known previously[1] and applies to more general objects than periodic points, i.e. normally hyperbolic invariant manifolds (NHIMs)

[1]Dellnitz, Junge, Lo, Marsden, Padberg, Preis, Ross, Thiere [2005] Phys. Rev. Lett.; Dellnitz, Junge, Koon, Lekien, Lo, Marsden, Padberg, Preis, Ross, Thiere [2005] Int. J. Bif. Chaos

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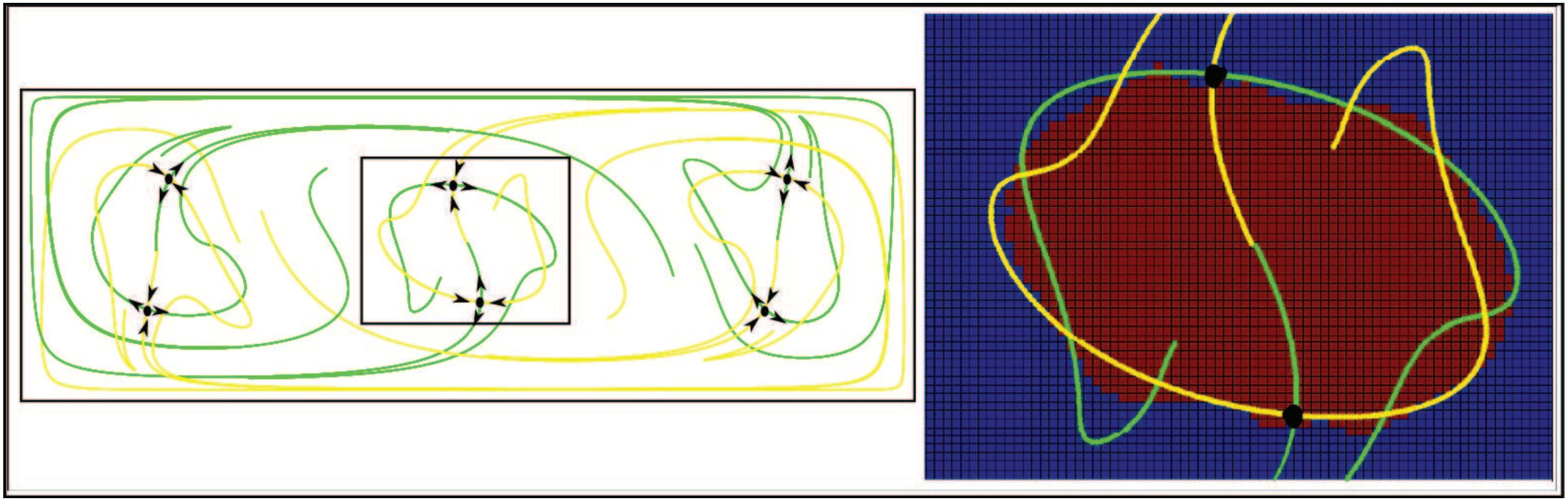

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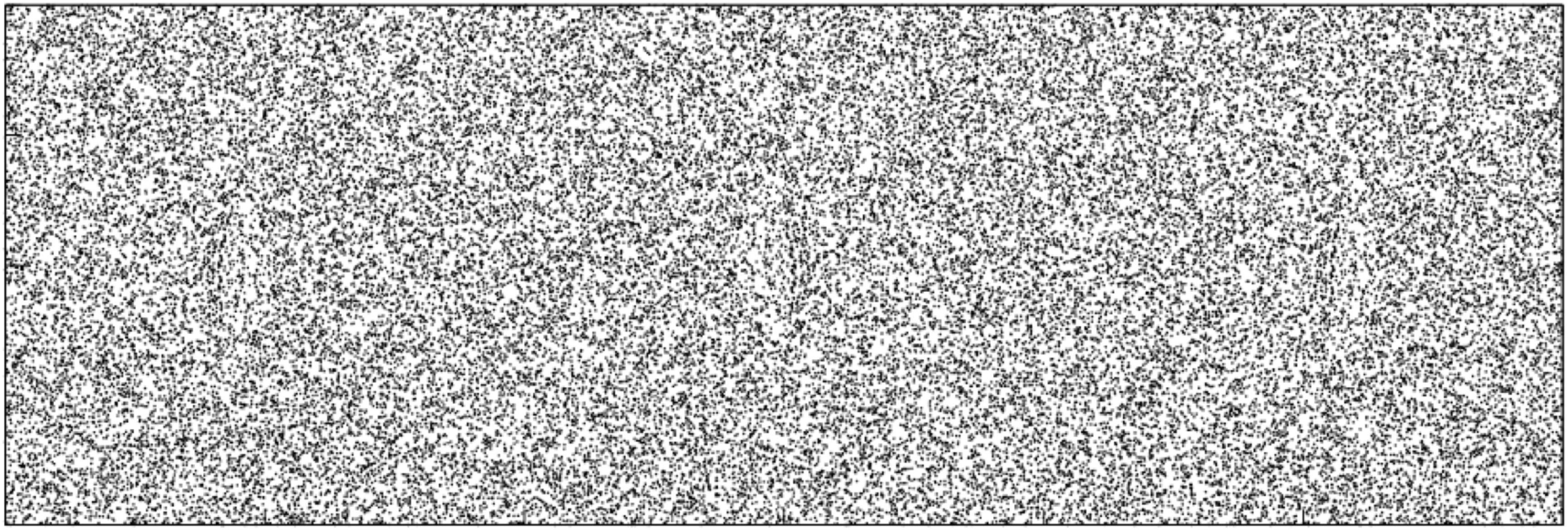


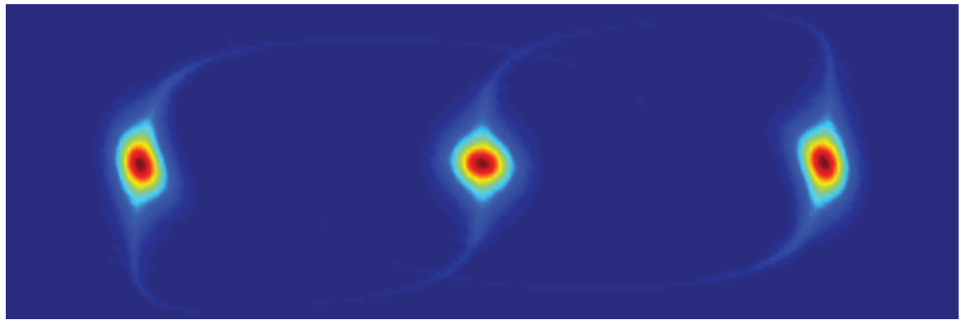
period-τ_f map for $\tau_f < 1 \Rightarrow$ no 'obvious' structure

- Return to $\tau_f < 1$ case, where no periodic orbits of low period known
- What are the AISs and ACSs here?

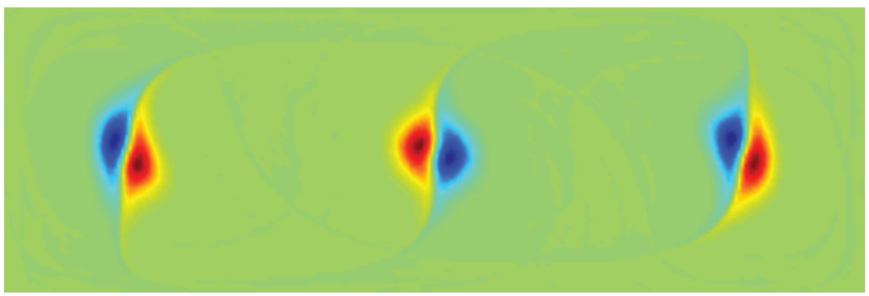
- Consider $P_t^{t+\tau_f}$ induced by family of period-τ_f maps $\phi_t^{t+\tau_f}$, $t \in [0, \tau_f)$

Identifying ‘ghost rods’: almost-cyclic sets

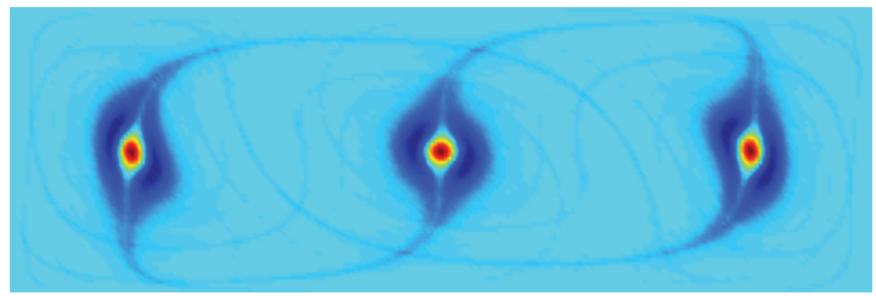
Top eigenvectors for $\tau_f = 0.99$ reveal hierarchy of phase space structures



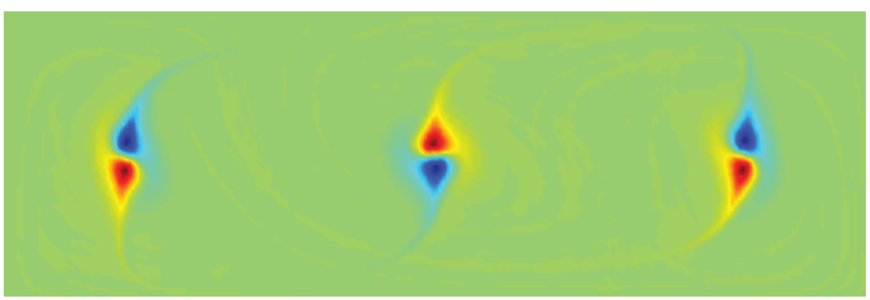
ν_2



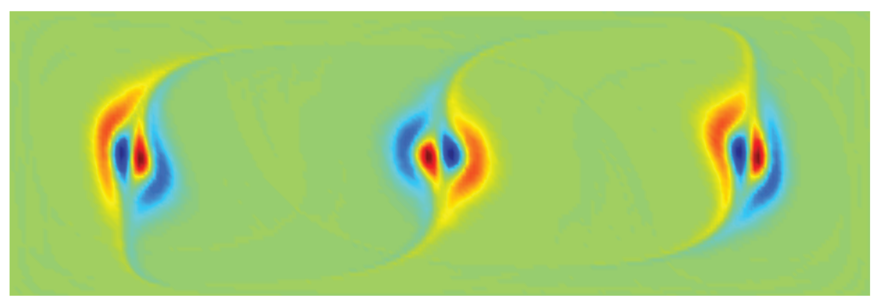
ν_3



ν_4

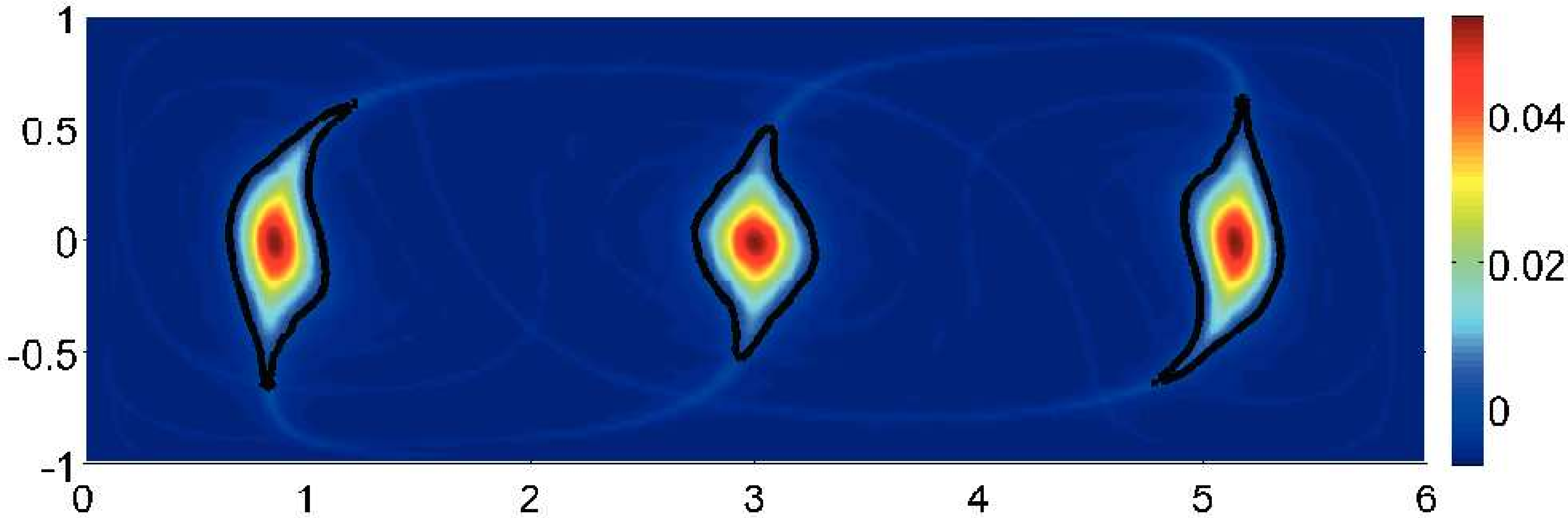


ν_5



ν_6

Identifying 'ghost rods': almost-cyclic sets



The zero contour (black) is the boundary between the two almost-invariant sets.

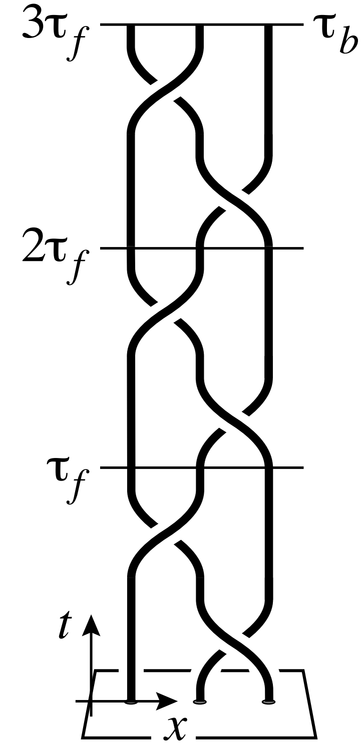
- Three-component AIS made of 3 ACSs of period 3
- ACS effectively replace periodic orbits for TNCT

Identifying ‘ghost rods’: almost-cyclic sets

Almost-cyclic sets stirring the surrounding fluid like ‘ghost rods’
— **works even when periodic orbits are absent!**

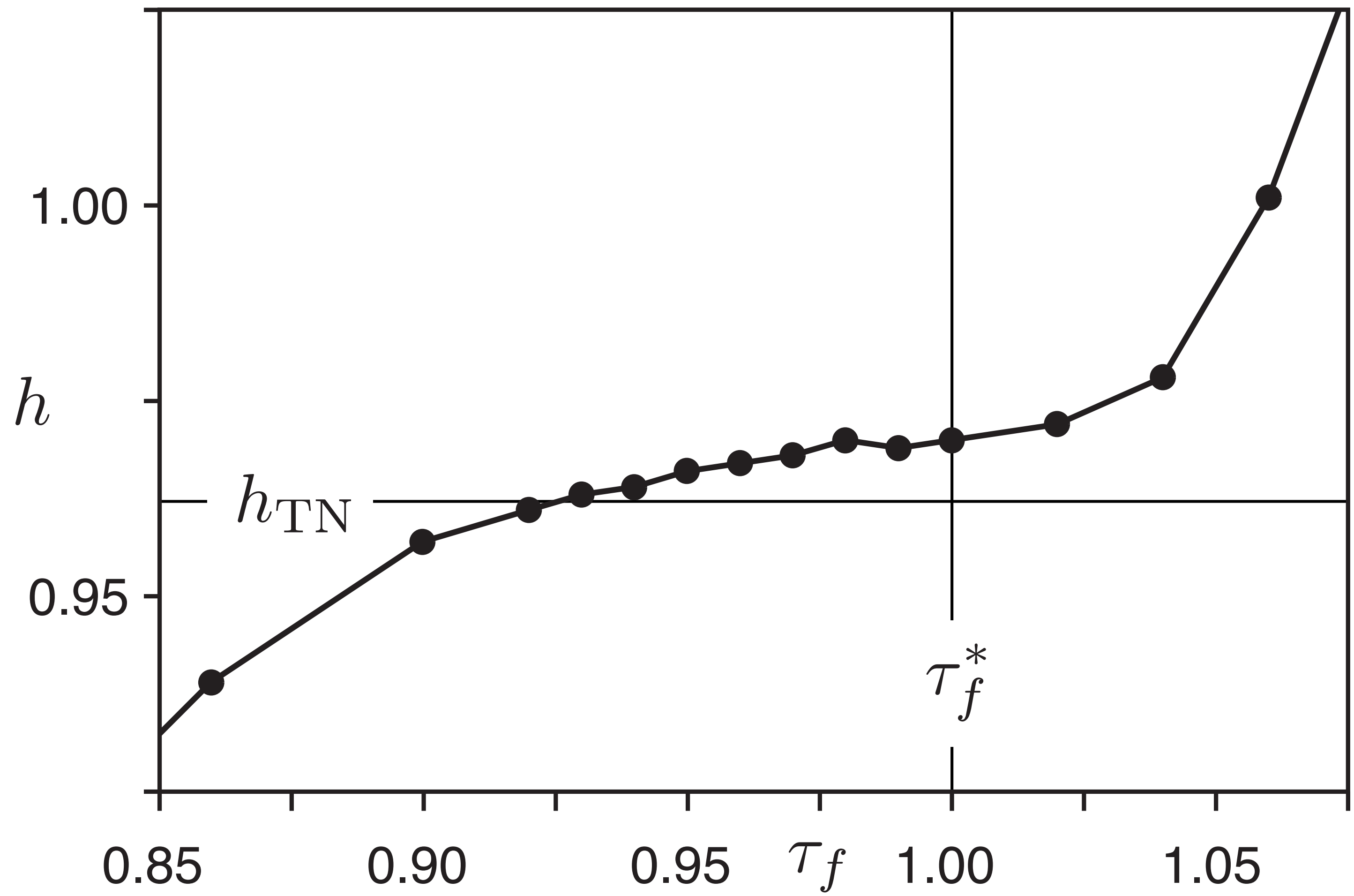
Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

Identifying 'ghost rods': almost-cyclic sets



Braid of ACSs gives lower bound of entropy via Thurston-Nielsen
— One only needs approximately cyclic blobs of phase space
— But, theorems apply only to periodic points
— Stremler, Ross, Grover, Kumar [2011] Phys. Rev. Lett.

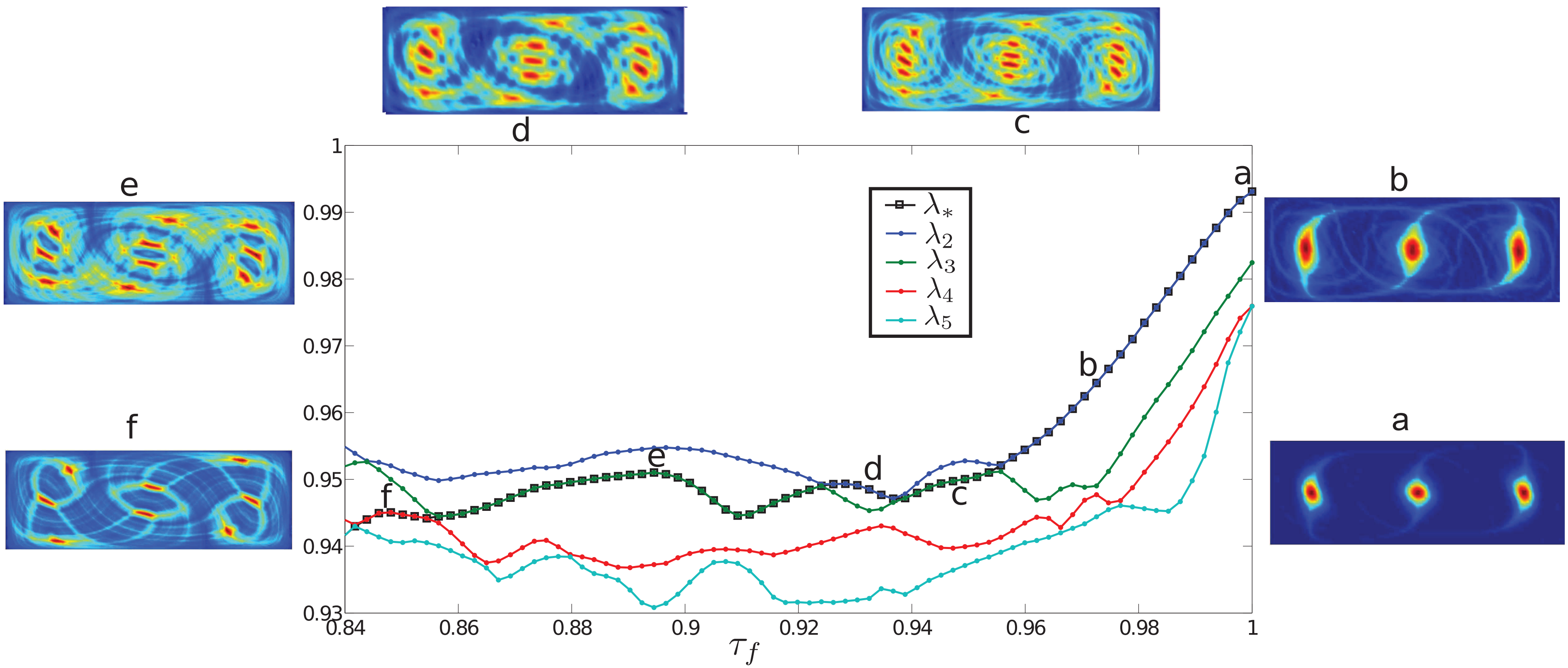
Topological entropy vs. bifurcation parameter



topological entropy as a function of τ_f

- h_{TN} shown for ACS braid on 3 strands

Eigenvalues/vectors vs. bifurcation parameter



Consider change in eigenvector[z] along continuous branch marked with '–□–' above (from a to f), as τ_f decreases $\Rightarrow$

[z]Inspired by Junge, Marsden, Mezic [2004]

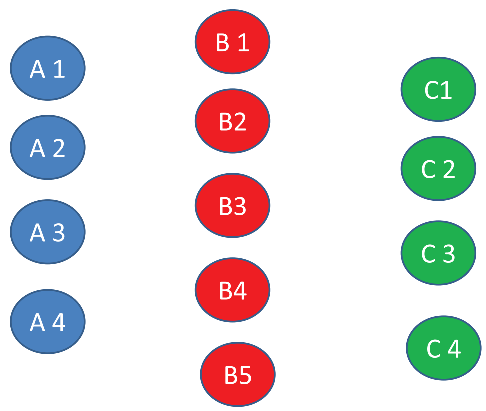
Bifurcation of ACSs — braid on 13 strands

For example, braid on 13 strands for $\tau_f = 0.92$

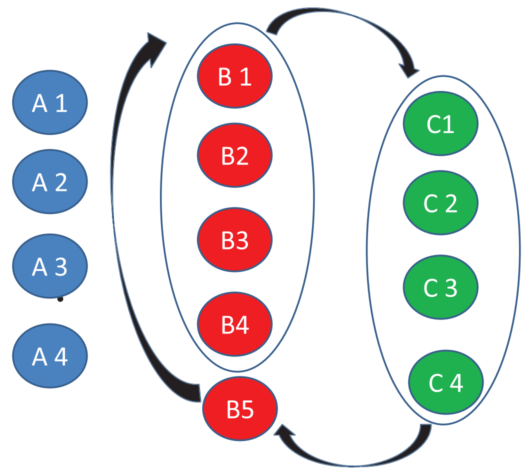
Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

Thurson-Nielsen for this braid provides lower bound on topological entropy

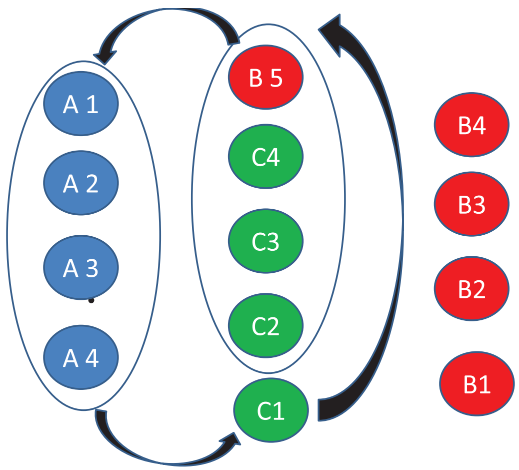
Bifurcation of ACSs — braid on 13 strands



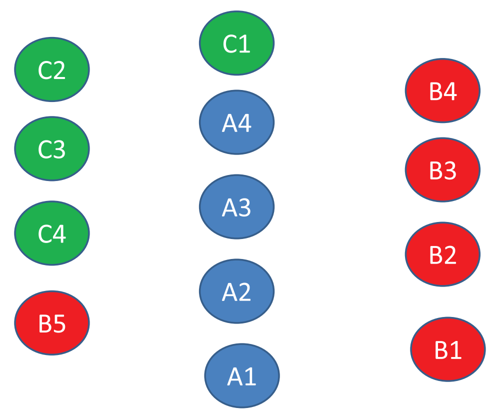
(a) Initial state



(b) First half-period

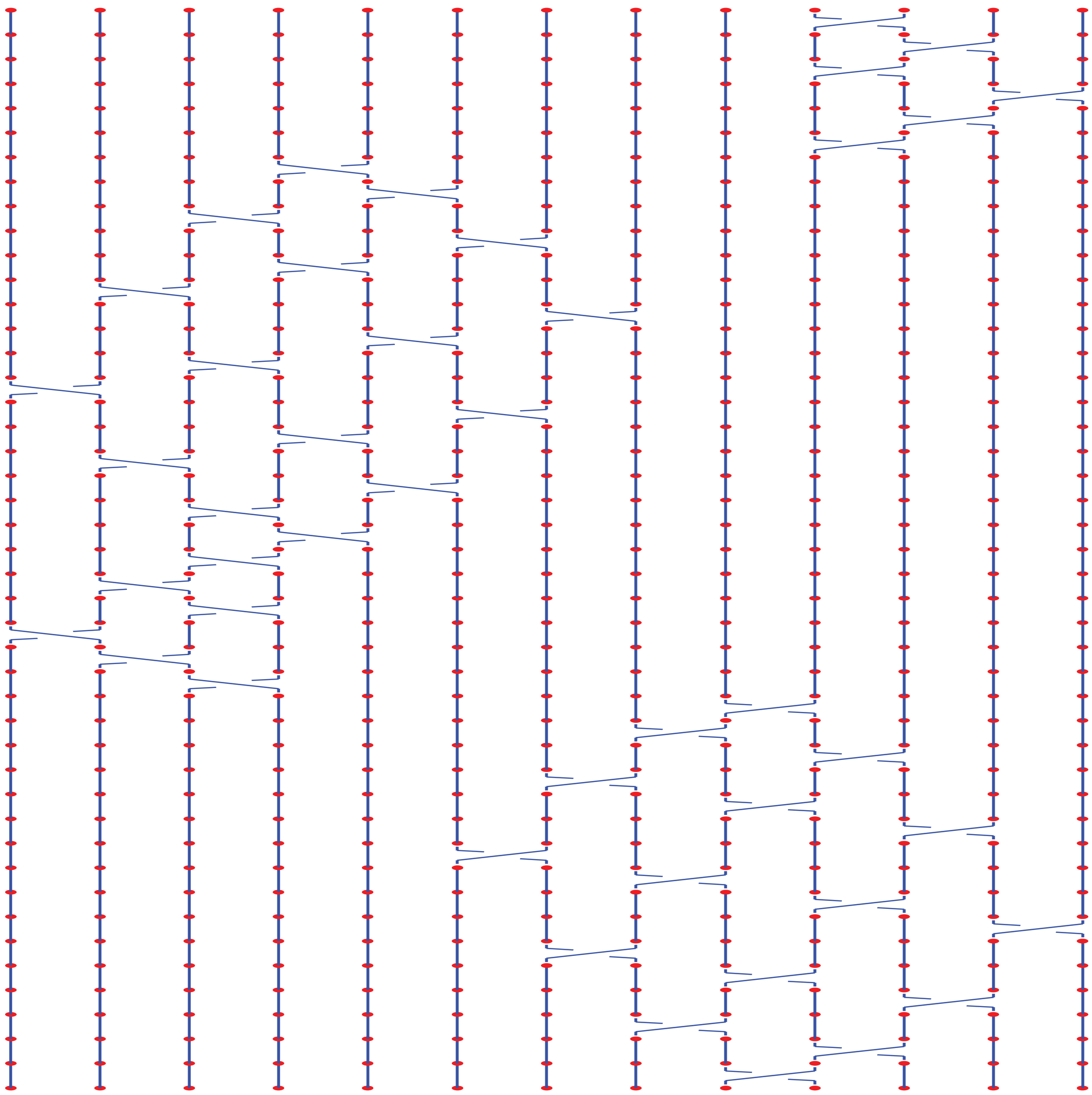


(c) Second half-period



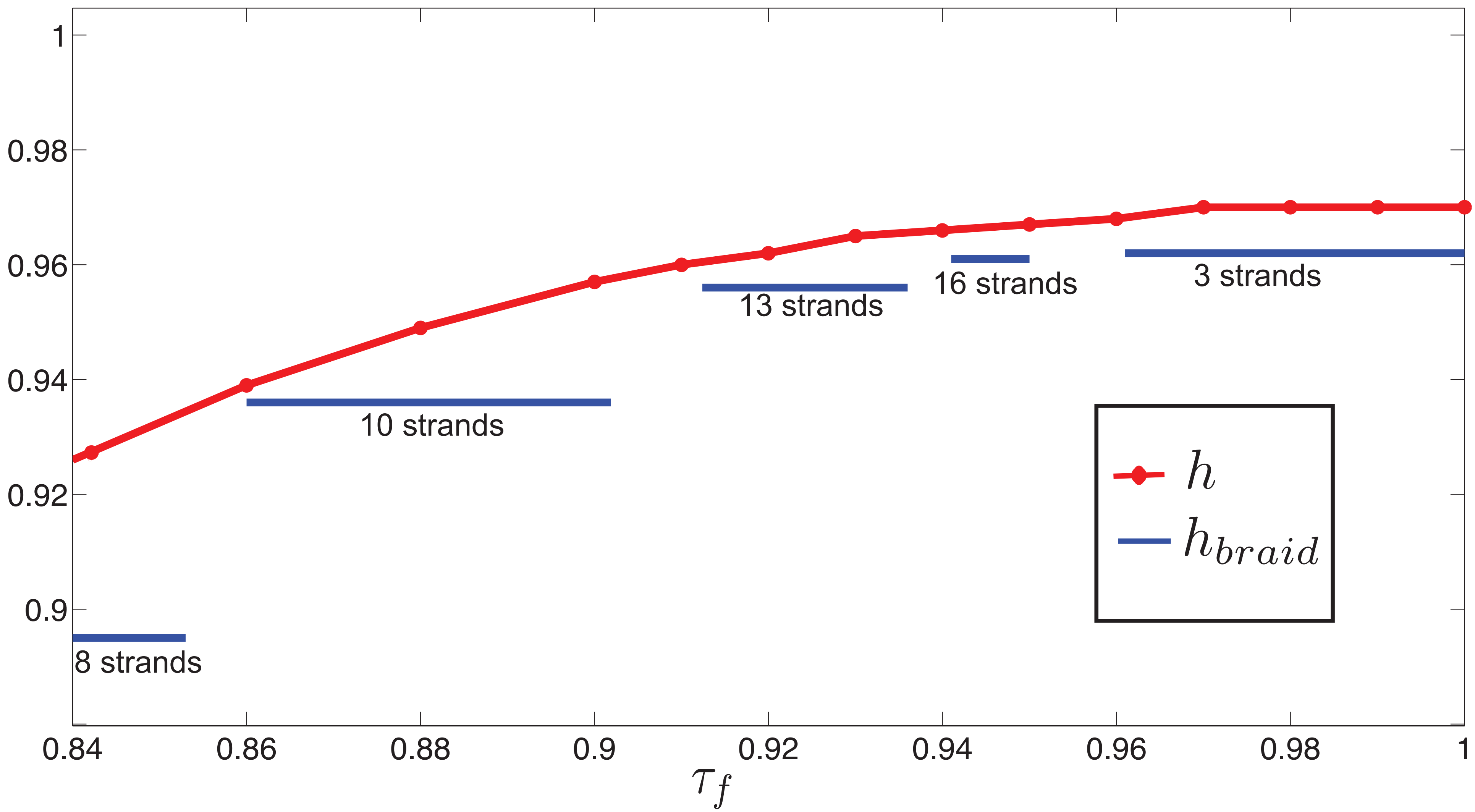
(d) State after 1 period

Bifurcation of ACSs — braid on 13 strands



representation of braid

Sequence of ACS braids bounds entropy



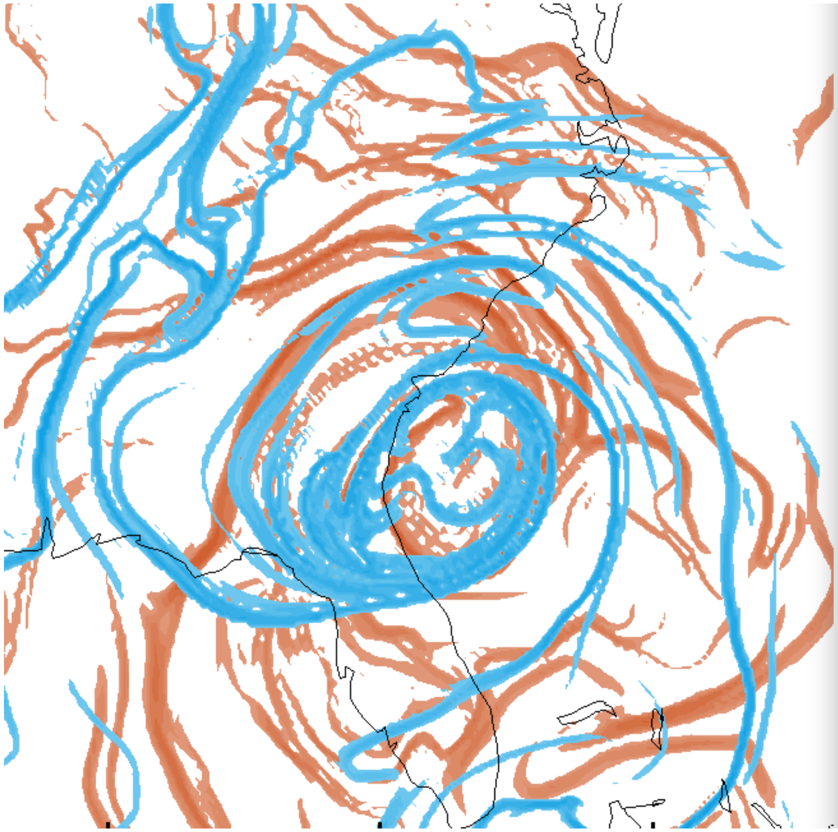
For various braids of ACSs, the calculated entropy is given, bounding from below the true topological entropy over the range where the braid exists

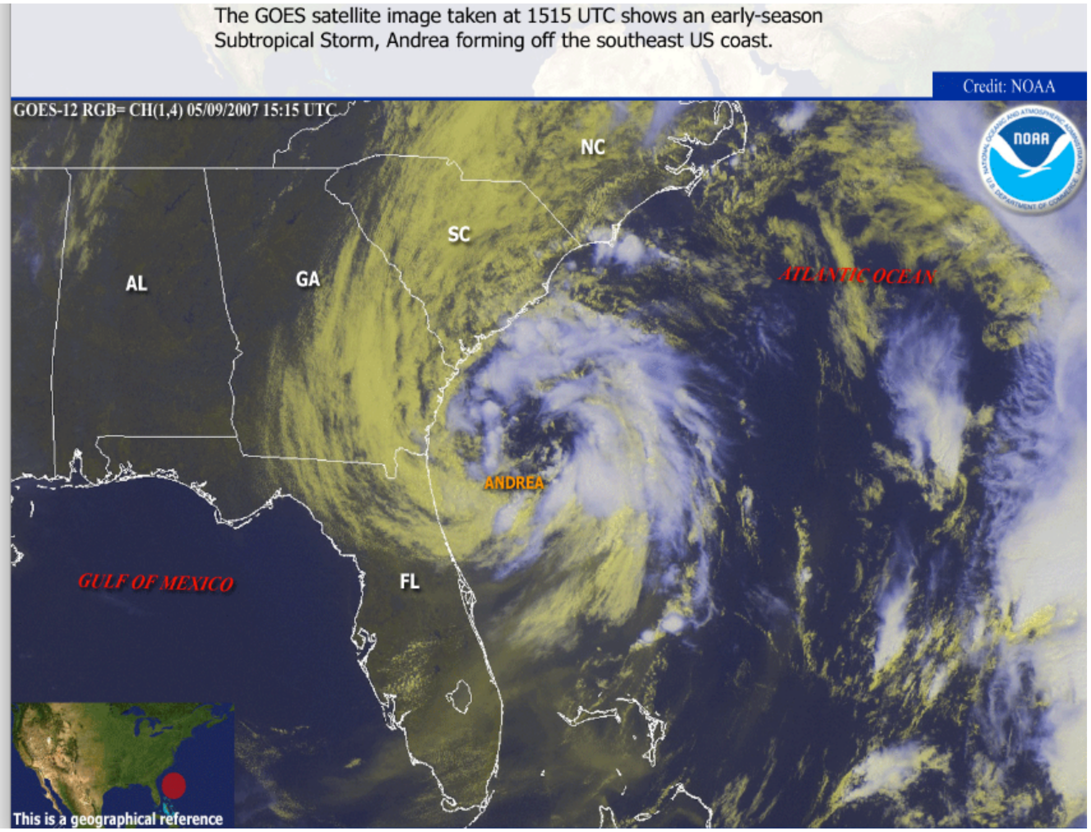
Non-autonomous, non-periodic, finite-time setting

- Data-driven, finite-time, non-periodic setting
  — e.g., from experimental fluid measurements, observations
- Are there, e.g., braids in realistic fluid flows?

LCSs: orange = repelling, blue = attracting

Atmospheric flows: hurricanes





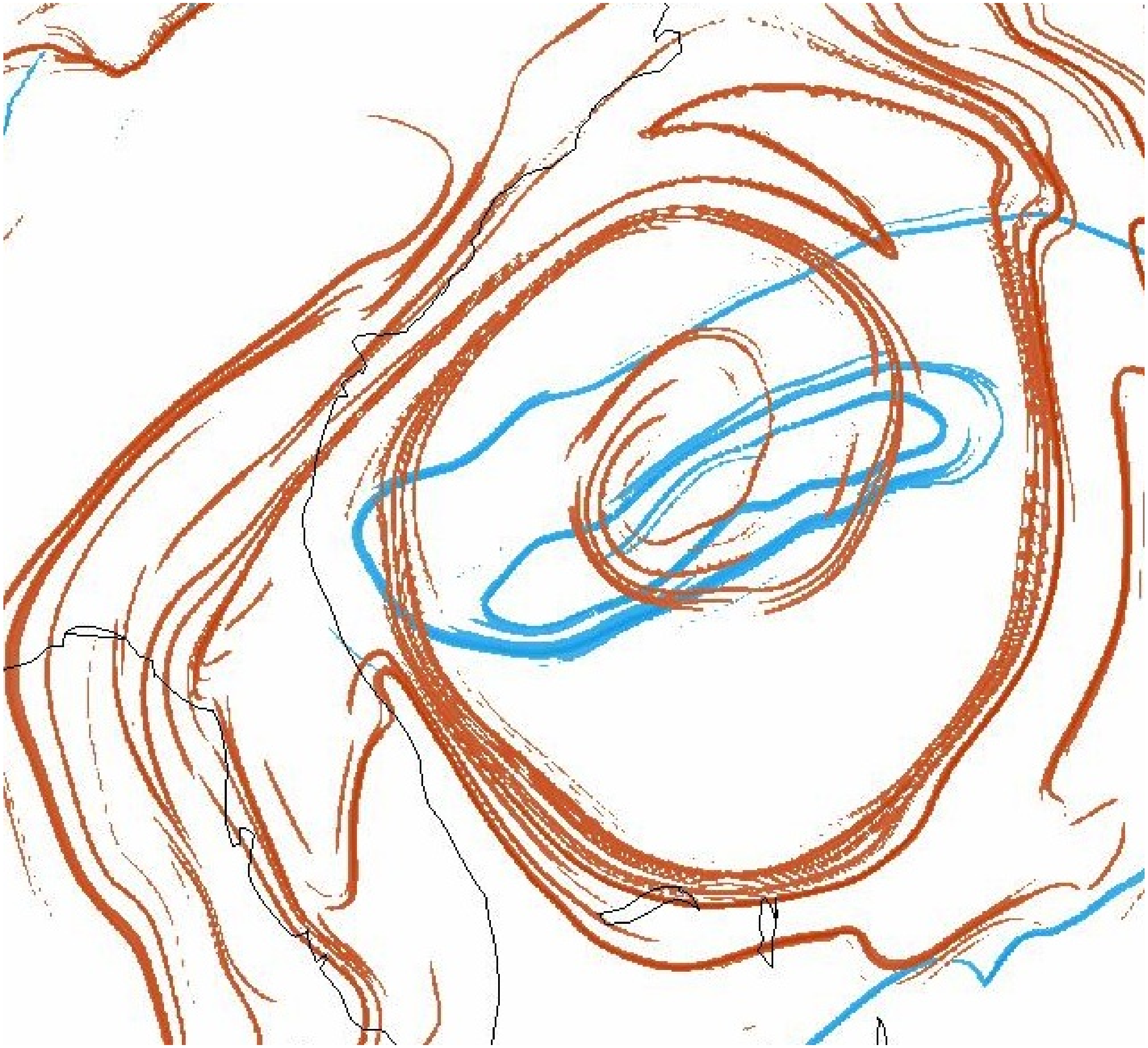
orange = repelling curves, blue = attracting curves

satellite

Andrea, first storm of 2007 hurricane season

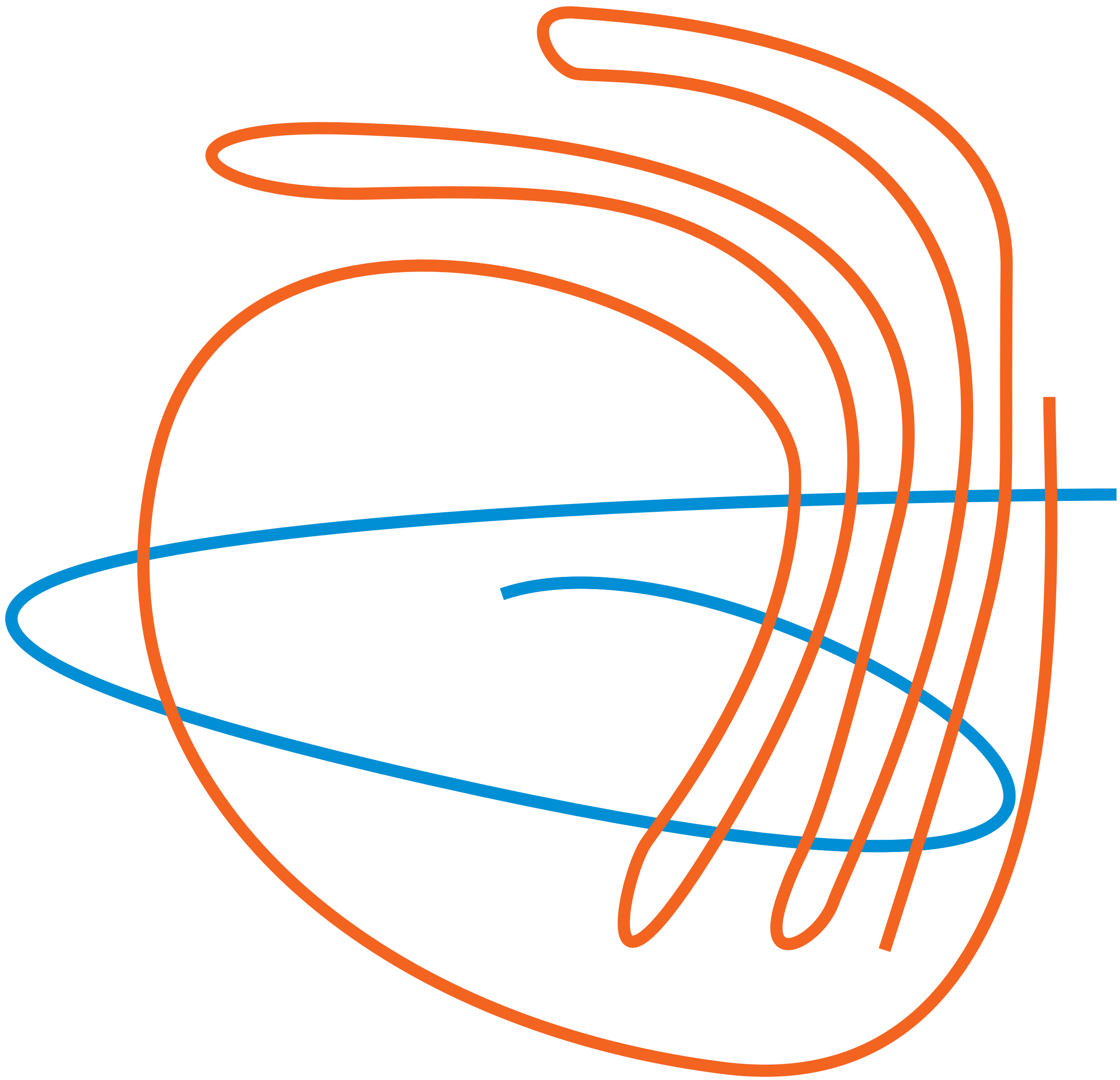
cf. Sapsis & Haller [2009], Du Toit & Marsden [2010], Lekien & Ross [2010], Ross &Tallapragada [2012]

Atmospheric flows: hurricanes



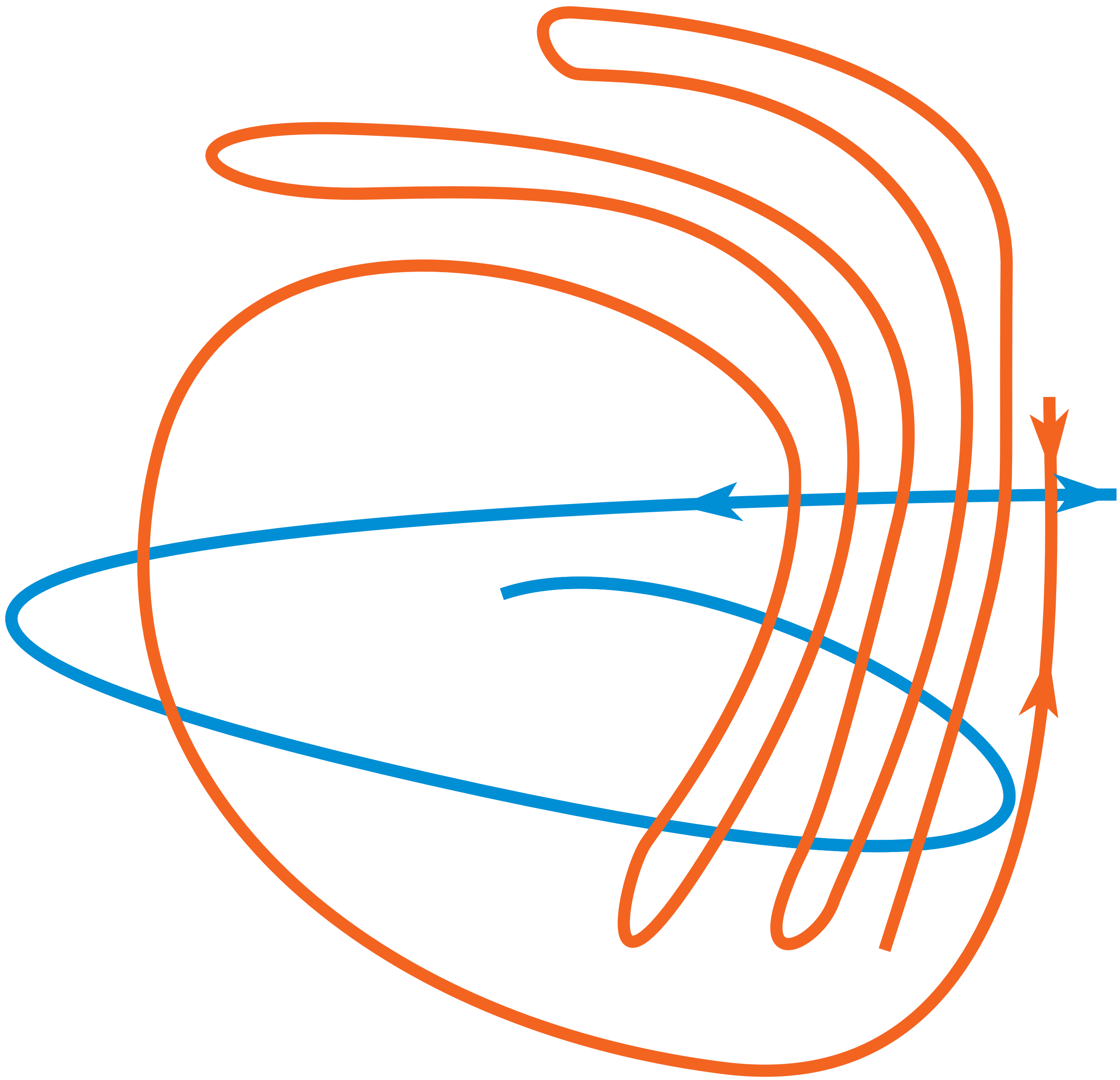
Andrea at one snapshot; Lagrangian coherent boundaries shown

Atmospheric flows: lobe dynamics to find braids



orange = repelling (stable manifold), blue = attracting (unstable manifold)

Atmospheric flows: lobe dynamics to find braids



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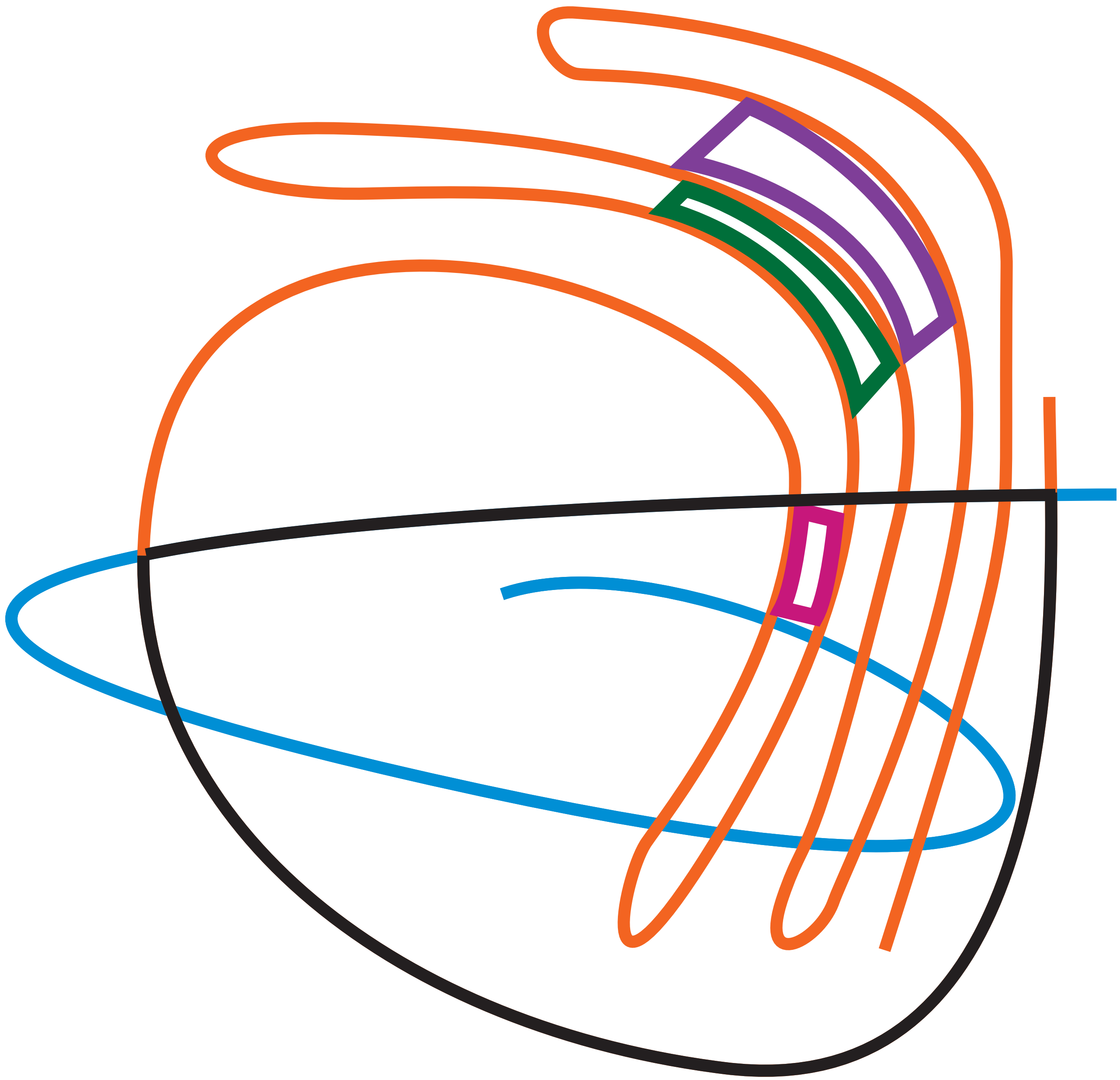
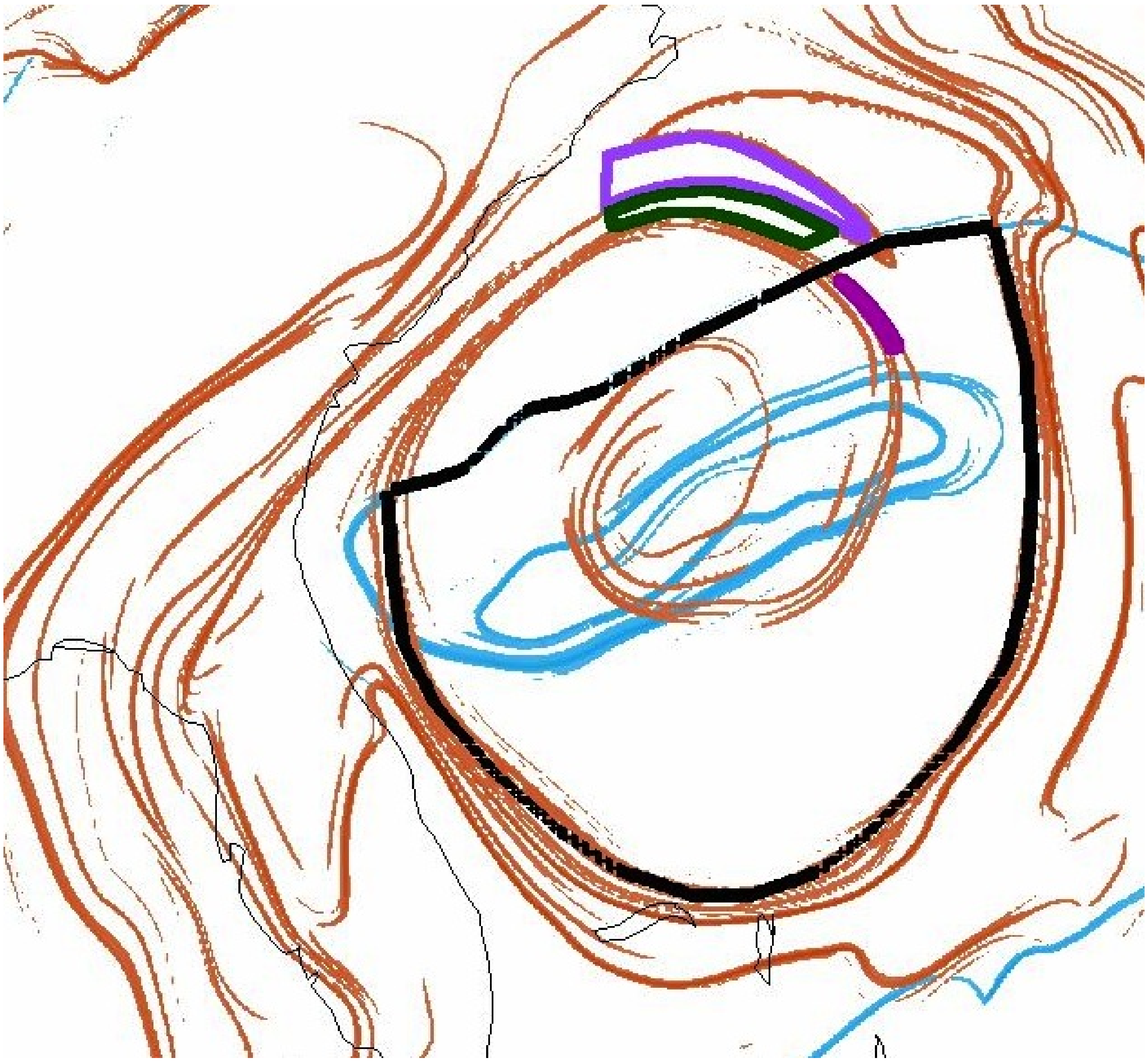


Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out

Atmospheric flows: lobe dynamics to find braids



Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out

Atmospheric flows: lobe dynamics to find braids

Sets behave as lobe dynamics dictates $\Rightarrow$ form braid, but no periodicity

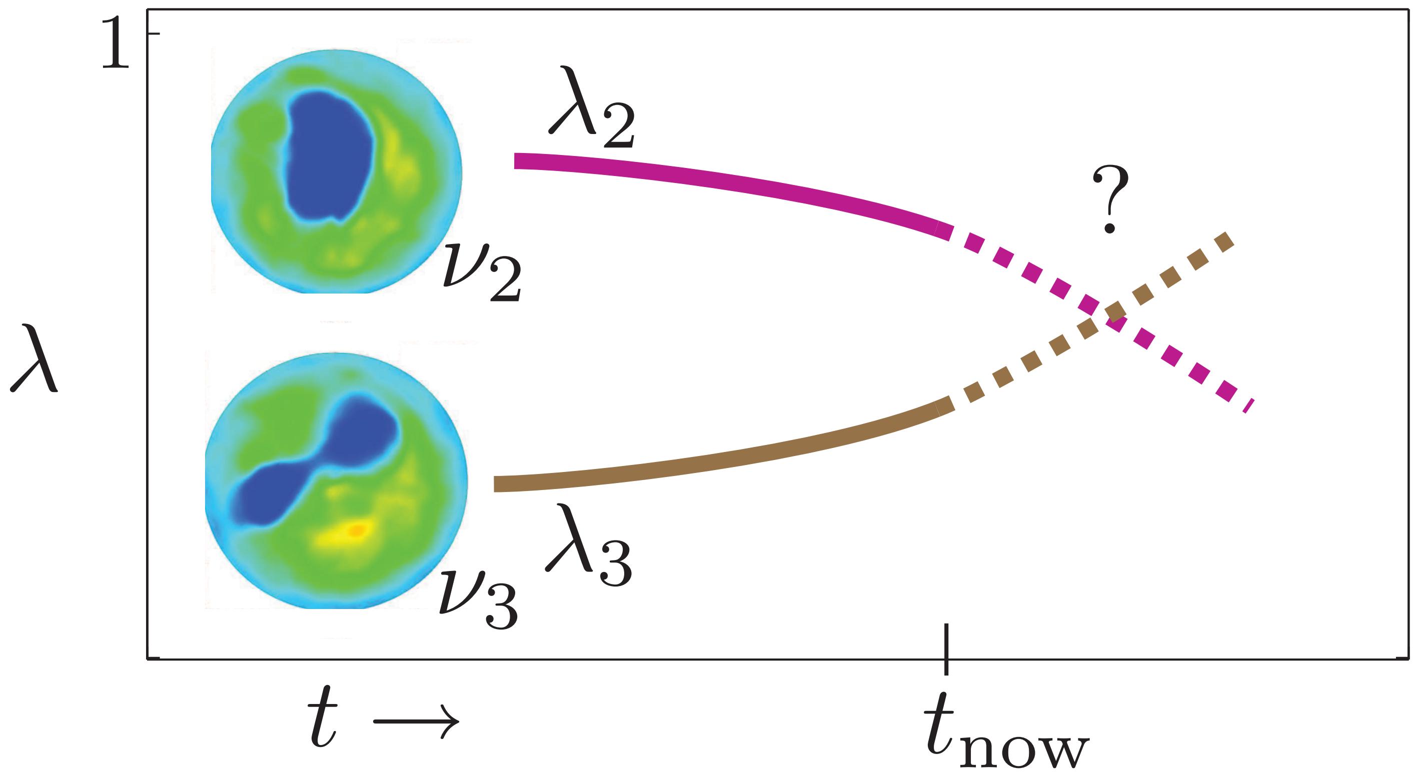
Atmospheric flows: Antarctic polar vortex

ozone data

Atmospheric flows: Antarctic polar vortex

ozone data + Lagrangian coherent boundaries (red = repelling, blue = attracting)

Speculation: trends in eigenvalues/vectors for prediction



- Different eigenvectors can correspond to dramatically different behavior.
- Some eigenvectors increase in importance while others decrease
- Can we predict dramatic changes in system behavior?
- e.g., splitting of the ozone hole in 2002, using only data *before* split

Final words

- Almost-cyclic sets enable application of the TNCT even in the absence of low-order periodic orbits.
  - For engineering systems, can design for mixing using ACSs
  - For natural systems, ghost rod/ACS paradigm may aid interpretation

- Connection between finite-time lobe dynamics and braids

- Bifurcation of phase space structure revealed through bifurcation of AIS/ACSs, braid bifurcations, etc.

- Prediction of dramatic changes in system behavior using changing order of eigenvectors?

The End

For papers, movies, etc., visit:
www.shaneross.com

Related Papers:

- Grover, Ross, Stremler, Kumar [2012] Topological chaos, braiding and bifurcation of almost-cyclic sets. Preprint.
- Stremler, Ross, Grover, Kumar [2011] Topological chaos and periodic braiding of almost-cyclic sets. *Physical Review Letters* 106, 114101.
- Lekien & Ross [2010] The computation of finite-time Lyapunov exponents on unstructured meshes and for non-Euclidean manifolds. *Chaos* 20, 017505.
- Senatore & Ross [2011] Detection and characterization of transport barriers in complex flows via ridge extraction of the finite time Lyapunov exponent field, *International Journal for Numerical Methods in Engineering* 86, 1163.
- Tallapragada & Ross [2008] Particle segregation by Stokes number for small neutrally buoyant spheres in a fluid, *Physical Review E* 78, 036308.