DYNAMICS OF LIBRATION POINT ORBITS IN THE EARTH-MOON-SUN SYSTEM

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Abstract. Orbits near libration points in the Earth-Sun system provide opportunities for investigations concerning submillimeter astronomy and terrestrial envirnments. Dynamics of these orbits can be approximated by periodic orbits in the planar circular restricted three-body problem. Invariant manifold theory is applied. Stable, unstable and center manifolds for libration points are approximated. Center manifold approximations are then used as the first step in a differential correction process to generate families of two-dimensional, periodic orbits near the collinear libration points L_1 and L_2. No stable orbits have been found. The invariant manifolds at L_1 and L_2 indicate homoclinic orbits may exist.

Introduction

The second Earth-Sun libration point, L_2 (see Fig. 1), is a quasi-stable equilibrium point in the gravitational potential 1.5 million km outside the Earth's orbit along the Earth-Sun line. It is an ideal location for an infrared astronomical mission because it provides a cold, stable environment and excellent viewing geometry. At any given time, more than half of the celestial sphere would be available for observation. The Far Infrared Explorer (FIRE) mission, under the direction of Professor Andrew Lange of Caltech, the principal investigator, is considering using this location to map the cosmic microwave background as a follow-on to the Cosmic Background Explorer (COBE) mission.

In theory, a point mass located at L_2 could remain there indefinitely. But due to perturbations and the dynamical instability of L_2, such a point mass will remain at L_2 for only a short time and will inevitably depart. However, there are period and quasi-periodic orbits in the neighborhood of L_2 known as Lissajous orbits. The International Sun-Earth Explorer (Farquhar *et al.*, 1977) was the first mission to use such an orbit, in the vicinity of the Sun-Earth interior libration point, L_1.

This study deals with the analytical and numerical development of a local approximation for two-dimensional, periodic orbits near the collinear libration points in the planar circular restricted three body problem. Dynamics of these orbits can be approximated numerically with the application of invariant manifold theory. Stable, unstable, and center manifolds for libration points were approximated. Center manifold approximations were then used as the first step in a differential correction process to generate families of two-dimensional, periodic orbits near the collinear libration points L_1 and L_2. No stable orbits have been found, due to the confinement of the approximations to the ecliptic plane.

The focus of current work is the development of numerical methods useful for determining trajectories to and from libration points and their associated periodic orbits. This can be accomplished through a strategy involving the stable and unstable invariant manifolds of points on halo orbits (Barden, 1994).

Equations of Motion

At present, the general problem of three bodies has no known analytic solution. The associated dynamical system has a phase space of 18 dimensions requiring 18 constants of integration to solve the differential equations. Only ten constants are known, thus a complete analytic solution is not possible. Therefore, assumptions are incorporated that simplify the general problem. The particular assumptions used lead to a model known as the planar circular restricted three-body problem (PCR3PB).

The PCR3PB involves two finite masses, m_1 and m_2, assumed to be point masses, moving respectively in circular motion around their common mass center, each under the gravitational influence of the other. A rotating coordinate system, with the origin at the common mass center is chosen (Fig. 2). The mass parameter μ is defined as the mass ratio m_2 to the sum $(m_1 + m_2)$.

The third body, m_3, is assumed to have infinitesimal mass relative to m_1 and m_2. The motion of m_3 is restricted to the x-y plane, defined by the circular motion of m_1 and m_2. For convenience, nondimensional units were chosen such that the following quantities are equal to unity: the distance between m_1 and m_2, the angular velocity of the rotating frame, and the sum of the two primary masses, $m_1 + m_2$.

In this system, there are five well known equilibrium solutions, or libration points, where gravitational and centrifugal forces acting on m_3 are balanced. With $m_1 > m_2$, the libration points are defined as shown in Fig. 2. In the nondimenisional rotating coordinate system, the equilateral points, L_4 and L_5, are located respectively at (.5 - μ, .5$\sqrt{3}$). The exact position of the collinear points depends on μ, but convention defines L_3 to be on the far side of m_1, L_1 to be between m_1 and m_2, and L_2 to be on the far side of m_2.

The equations of motion for the system, as derived in Szebehely (1967), are:

$$\ddot{x} = \frac{\partial U}{\partial x} + 2\dot{y},$$
$$\ddot{y} = \frac{\partial U}{\partial y} - 2\dot{x}. \qquad (1)$$

where the pseudo-potential, U, takes the form

$$U = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)}{d_1} + \frac{\mu}{d_2};$$
$$d_1 = \sqrt{(x+\mu)^2 + y^2};$$
$$d_2 = \sqrt{(x+\mu-1)^2 + y^2};$$
$$\mu = m_2;$$
$$1 - \mu = m_1.$$

This system does admit a constant of integration (C) identified by the Jacobi which takes the form

$$C = 2U - (\dot{x}^2 + \dot{y}^2). \qquad (2)$$

The Jacobi constant is extremely valuable as a means to check the accuracy of numerical integration.

Application of Invariant Manifold Theory

The first procedure in the numerical construction of collinear libration point orbits is to develop an approximate 'first guess' trajectory which can then be refined through a process of differential correction. Techniques previously used to approximate periodic orbits about the collinear points

have involved analytical development (Richardson, 1979). Invariant manifold theory provides an exceptionally elegant method to numerically approximate such orbits. The discussion that follows presents invariant manifold theory in the context of the PCR3BP.

The equations of motion (1), can be written as a nonlinear vector system defined on $\Re^4$

$$\dot{x} = f(x). \tag{3}$$

Let $\bar{x}$ be a fixed point of the nonlinear system (i.e. libration point), such that

$$\dot{x} = f(\bar{x}). \tag{4}$$

By the discussion in Wiggins (1990), it is reasonable to consider the associated *linear* system

$$\dot{y} = A\,y. \tag{5}$$

where $A \equiv Df(\bar{x})$ is a constant 4 x 4 matrix. Now $\Re^4$ can be represented as the direct sum of three embedded subspaces (manifolds) denoted E^s, E^u, and E^c, which are defined as follows:

$$\begin{aligned} &\text{(stable)} \quad E^s = \text{span}\{e_1, \cdots, e_s\}, \\ &\text{(unstable)}\ E^u = \text{span}\{e_{s+1}, \cdots, e_{s+u}\}, \qquad s+u+c = 4, \\ &\text{(center)} \quad E^c = \text{span}\{e_{s+u+1}, \cdots, e_{s+u+c}\}, \end{aligned} \tag{6}$$

where $\{e_1, \cdots, e_s\}$ are the eigenvectors of A corresponding to the eigenvalues having *negative real part*, $\{e_{s+1}, \cdots, e_{s+u}\}$ are the eigenvectors of A corresponding to the eigenvalues having *positive real part*, and $\{e_{s+u+1}, \cdots, e_{s+u+c}\}$ are the eigenvectors of A corresponding to the eigenvalues having *zero real part*. Solutions starting in E^s approach $\bar{x}$ as $t \to +\infty$ and solutions starting in E^u approach $\bar{x}$ as $t \to -\infty$. Solutions starting in E^c tend neither toward nor away from $\bar{x}$ as $t \to \pm\infty$.

Any libration point can be taken for $\bar{x}$. Thus, from Eqs. (5) and (6), we have a linear approximation for the invariant manifolds within a sufficiently small neighborhood of a libration point. One can expand the local approximation to the entire phase space surrounding a libration point by globalizing the manifolds.

Near the libration point $\bar{x}$, any of the manifolds E^s, E^u, or E^c, can be approximated by their respective eigenvectors. Thus, the set of eigenvectors, e_m, associated with a particular manifold can be used to obtain an approximation for an inital state vector, $\bar{x}_0$, which can serve as an initial condition to globalize the manifold. Globalization can be performed by numerically integrating the equations of motion (Eqs. (1)) forward (and backward) in time from the starting point, $\bar{x}_0$. To determine $\bar{x}_0$, the coordinates for the position are displaced from the libration point L_n in the direction of by some scalar γ. The coordinates for the velocity are also in the direction of e_m multiplied by γ. The initial state vector is thus defined as

$$\bar{x}_0 = L_n + \gamma e_m \tag{7}$$

The magnitude of γ should be small enough to avoid violating the legitimacy of the linear approximation, but not so small that errors associated with numerical integration can arise.

For the collinear points, the calculation of the stable and unstable manifolds is straight forward; there is one eigenvector associated with the stable manifold and one with the unstable manifold. Due to the nature of center manifolds however, there are two associated complex eigenvectors. Thus, to approximate the center manifold, we need some new definitions.

The eigenspace of the center manifold, E^c, is defined as

$$E^c = \text{span}\{e_{c1}, e_{c2}\} \tag{8}$$

To obtain a real value, we use the projection of this eigenspace on the unit circle, defined as follows

$$\hat{E}_{c1} = \frac{e_{c1} + e_{c2}}{|e_{c1} + e_{c2}|} \quad , \quad \hat{E}_{c2} = \frac{i(e_{c1} - e_{c2})}{|e_{c1} - e_{c2}|}$$
$$E_c = a\hat{E}_{c1} + b\hat{E}_{c2} \quad , \quad a^2 + b^2 = 1. \tag{9}$$

The values of a and b can be varied if necessary. We can now define an initial state vector,

$$\bar{x}_0^c = L_n + \gamma E_c \tag{10}$$

from which the center manifold of a libration point can be globalized. The vector $\bar{x}_0^c$ can be used as an initial guess to generate families of two-dimensional, periodic orbits.

Differential Correction Process

Differential correction procedures play an important role in identifying particular initial conditions which result in motion producing orbits. This requires the state transition matrix, which reflects the sensitivity of the state at time t to small perturbations in the initial state at time t_0. Equations (1) can be rewritten as four first order differential equations, from which a state column vector can be defined as $\bar{x} = [x\ y\ \dot{x}\ \dot{y}]^T$. Given some known solution, $\bar{x}(t)$, to equations (1), the differential equation for the state transition matrix $\Phi(t, t_0)$, the matrix of partial derivatives $\partial\, \bar{x}(t) / \partial\, \bar{x}(t_0)$ associated with these equations of motion, is

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \tag{11}$$

where $A(t)$ is a 4 x 4, (generally) time varying matrix which is divided into four 2 x 2 submatrices

$$A(t) = \begin{bmatrix} 0 & I_2 \\ U_{nn} & 2\Omega \end{bmatrix}, \tag{12}$$

where 0 represents the zero matrix and I_2 is the 2 x 2 identity matrix. The matrix Ω can be written

$$\Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{13}$$

and U_{xx} has the form

$$U_{nn} = \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix}, \quad \text{where} \quad U_{ab} = \frac{\partial^2 U}{\partial a \partial b}. \tag{14}$$

The initial condition $\Phi(t_0, t_0)$ is equal to the 4 x 4 identity matrix. This gives 16 first order scalar differential equations representing the elements of the state transition matrix. Combined with the four first order scalar equations of motion, the problem thus presents a set of 20 differential equations requiring simultaneous numerical integration.

The algorithm below closely follows that of Howell (1984) and Breakwell and Brown (1979) and adapts it to the PCR3BP. Note that the system of equations (1) are invariant under the transformation $y \rightarrow -y$ and $t \rightarrow -t$. For simplification, let an initial vector be a center manifold approximation of the form

$$\bar{x}_0^c = x_0 = [x_0\ \ 0\ \ 0\ \ \dot{y}_0]^T \tag{15}$$

which is perpendicular to the x axis. Since the periodic solution about a collinear point will be symmetric with respect to the x axis, if another perpendicular crossing can be found, such that

$$x_1 = [x_1 \;\; 0 \;\; 0 \;\; \dot{y}_1]^T, \tag{16}$$

then the orbit will be periodic with period $T = 2t_1$.

The transition matrix $\Phi(t_1, 0)$ at the end of a half-cycle of a nearly periodic orbit can be used to adjust the initial conditions so as to obtain periodicity. Using a numerical integration procedure, the equations are integrated until y changes sign. The accuracy of the integration is increased and the integration proceeds again. This process is repeated, until $|y| \leq 10^{-11}$, and the time at this point is defined to be t_1. The orbit is considered periodic if $|\dot{x}| \leq 10^{-10}$ at t_1. If this is not the case, $\dot{x}$ can be reduced by correcting the initial velocity $\dot{y}_0$ and integrating again.

Assume $|\dot{x}|$ is too large. The desired correction to our initial conditions is $\delta\dot{y}_0$. Since x_0 is held fixed, the correction can be calculated from

$$\delta\dot{x}_1 \cong \Phi_{34}\delta\dot{y}_0 + \ddot{x}_1\delta t, \tag{17}$$

where

$$0 = \delta y_1 = \Phi_{24}\delta\dot{y}_0 + \dot{y}_1\delta t, \tag{18}$$

and where Φ_{ij} is a typical element of $\Phi(t_1, 0)$. Hence

$$\delta\dot{y}_0 \cong \left(\Phi_{34} - \frac{1}{\dot{y}_1}\Phi_{24}\ddot{x}_1\right)^{-1}\delta\dot{x}_1, \tag{19}$$

and $\delta\dot{y}_0$ produces a $\delta\dot{x}_1$ that cancels out any previous $\dot{x}_1$. This provides an iterative calculation of $\dot{y}_0$. Using this methodology, the convergence to a periodic orbit occurs within three to four iterations.

Integration of the second half of the orbit can be performed to obtain the transition matrix at the end of a full cycle of the periodic orbit,

$$\Phi(T, 0) = \Phi(2t_1, 0). \tag{20}$$

It's eigenvalues determine the first order stability of the orbit; two of the eigenvalues are always 1 and the other two are a self-reciprocal set $(\lambda, 1/\lambda)$ which follows from the invariance of (1) under $y \rightarrow -y$ and $t \rightarrow -t$. The periodic orbit is thus unstable unless λ has a modulus of 1.

Numerical Results

A few hundred orbits were calculated to produce the results that follow. This relatively large number was necessary since the initial guess for an orbit around a particular collinear point was obtained from the center manifold approximation for the point. The study thus started with an orbit very close to the point and began each successive orbit further from the point, along the x axis. The study was performed for the Earth-Sun case ($\mu = 3.03591 \times 10^{-6}$).

The eigenvalues $(\lambda, 1/\lambda)$ of $\Phi(T, 0)$ have been computed, along with the periodic orbits themselves, following the process described above. The xy projections of the orbits are shown in Fig. 3. Stability is conveniently decided by the values of stability index

$$\nu = \tfrac{1}{2}[\lambda + (1/\lambda)] \tag{21}$$

The orbit is stable if $|\nu| \leq 1$.

Conclusions

As of this writing, no data are available in a presentable form. Further computer time is required to verify solutions and confirm some suspicions concerning the nature of those solutions. Results will be available by early November, 1995.

References

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