Experimental

Experimental verification of criteria for transition between potential wells in a 2 DOF system

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Our objective is to study routes of escape from a potential well in a 2 degree of freedom system. For this aim we study the chaotic motion of a rolling ball on a surface ($H(x,y)$) which has 4 potential wells, one in each quadrant of the (x,y) plane, Fig. 1(a)(b) [1]. We adopt a global geometric view of the motion analysis, using techniques which have been fruitful in other areas of mechanics, such as celestial mechanics [2] and physical chemistry [3]. The equations of motion are obtained from the Lagrangian; $\mathcal{L}(x,y,\dot{x},\dot{y}) = T(x,y,\dot{x},\dot{y}) - U(x,y)$. The kinetic energy (translational plus rotational for a ball rolling without slipping) is,

$$T(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}\tfrac{7}{5}\left(\dot{x}^2 + \dot{y}^2 + (H_x\dot{x} + H_y\dot{y})^2\right) \tag{1}$$

and the potential energy is $U(x,y) = gH(x,y)$, where,

$$H(x,y) = \alpha(x^2 + y^2) - \beta(\sqrt{x^2 + \gamma} + \sqrt{y^2 + \gamma}) - \xi xy + H_0. \tag{2}$$

We use parameter values $(\alpha, \beta, \gamma, \xi) = (0.07, 1.017, 15.103, 0.00656)$ in the appropriate units, along with $H_0 = 12.065$ cm and $g = 981$ cm/s^2. The ball's mass factors out so we do not include it.
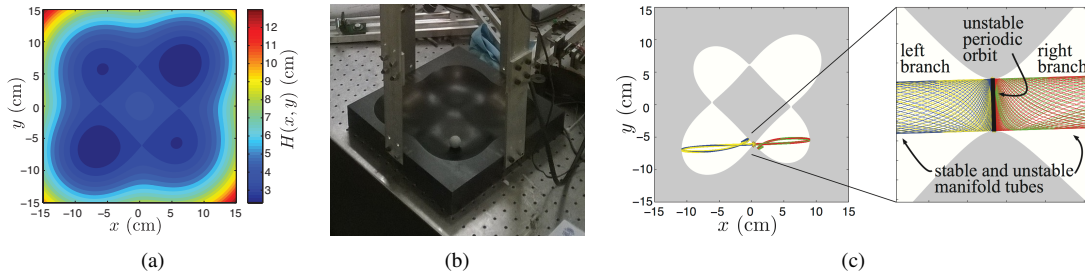

Figure 1. (a) Surface height $H(x,y)$. (b) Experimental apparatus. (c) For a fixed energy, E, above a critical value E_e, the permissible region (in white) has potential wells connected by necks around saddle-type equilibria. All motion between adjacent wells must occur through the interior of the stable and unstable manifold tubes associated to an unstable periodic orbit in the neck.

Small damping is present, but over short time-scales, the motion approximately conserves energy, and the conservative dynamics are the dominant contributor to transition between wells. Let $\mathcal{M}$ be the *energy manifold* given by setting the energy integral ($\mathcal{E}(x,y,v_x,v_y) = T(x,y,\dot{x},\dot{y}) + U(x,y)$) equal to a constant, i.e., $\mathcal{M}(E) = \{(x,y,v_x,v_y) \subset \mathbb{R}^4 \mid \mathcal{E}(x,y,v_x,v_y) = E\}$ where E is a constant. The projection of the energy manifold onto configuration space, the (x,y) plane, is the region of energetically possible motion for a ball of energy E, $M(E) = \{(x,y) \mid U(x,y) \leq E\}$. The zero velocity curves are the boundary of $M(E)$ and are the locus of points in the (x,y) plane where the kinetic energy vanishes. The ball's state is only able to move on the side of this curve where the kinetic energy is positive, shown in white in Fig. 1(c). The

critical energy of escape, E_e, is the same as the energy of the saddle points in each neck (which are all equal), and divides the global behavior into two cases, according to the sign of $\Delta E = E - E_e$:

Case 1, $\Delta E < 0$: the ball is safe against escape since potential wells are not energetically connected.

Case 2, $\Delta E > 0$: "necks" between all the potential wells open up around the saddle points, permitting the ball to move between the two potential wells (e.g., Fig. 1(c) shows this case).

Tubes Leading to Escape. Within a given potential well the set of all states leading to escape to a different potential (or having just escaped a different potential well) are within a cylindrical manifold or *tube*, as in Fig. 1(c), where only the configuration space projection is shown. This tube is the set of all states for a fixed energy which will soon reach, or have just passed from, a different potential well [2, 3] (nested energy manifolds will have correspondingly nested tubes). For each E, the boundary of the tubes in phase space (or more precisely, within $\mathscr{M}(E)$) are the stable and unstable manifolds of an unstable periodic orbit of the same energy residing in the neck connecting the adjacent wells.

Experiment Results. We use two Poincaré sections which are selected based on the symmetry of the surface and the equations of motion, and which are best described in polar coordinates (r, p_r); $U_1^\pm = \{(r, p_r) \mid \theta = \frac{\pi}{2}, \text{ sign}(p_\theta) = \mp 1\}$ and $U_2^\pm = \{(r, p_r) \mid \theta = \frac{3\pi}{2}, \text{ sign}(p_\theta) = \mp 1\}$. Taking Poincaré sections of the experimental trajectories (e.g., the typical trial shown in Fig. 2(a)) should reveal the tube cross-sections. We first determine the instantaneous ΔE for every point on the Poincaré section, so we can consider only narrow ranges of ΔE, to approximate a single energy manifold, Fig. 2(b). In Fig. 2(c) we see an example of the Poincaré section U_1^+ and the points inside and outside of tubes identified by different colors for the range of energy $0 < \Delta E < 100 \ (\text{cm/s})^2$. The intersection points which have either just transitioned from one well or are about to transition to another, determined by following the trajectory backward and forward in time, are colored red and magenta, respectively. The clustering of the transitioning points shows good agreement with theory. Furthermore, the fraction of transitioning trajectories increases linearly with ΔE, Fig. 2(d), as expected theoretically from arguments related to the phase space flux over a saddle [4].
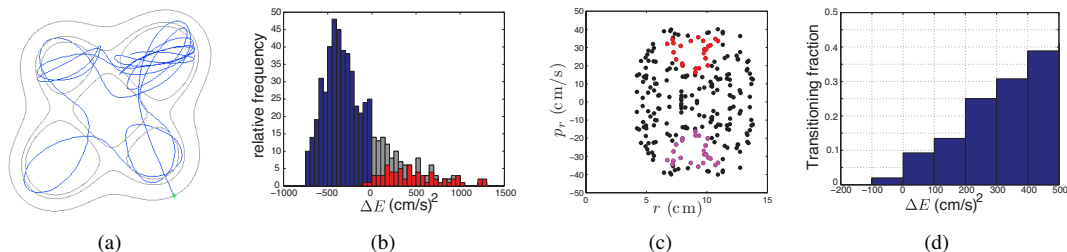

(a)   (b)   (c)   (d)

Figure 2. *Experimental results.* (a) A typical experimental trajectory in blue, which is released from rest at the lower right. Notice how it moves seemingly erratically between the 'wells'. Some height iso-contours are also shown, in light gray. (b) Histogram of energy for crossings of the U_1^+ Poincaré section (blue: $\Delta E < 0$, gray: $\Delta E > 0$, red: transitioning). (c) On U_1^+, we consider only a narrow range of energy ($\Delta E \in [0, 100 \ (\text{cm/s})^2]$) and label intersecting trajectories by their recent past or future; black: no transition, red: recent transition to quadrant 1 from quadrant 2 and magenta: imminent transition from quadrant 1 to 4. (d) Fraction of transitioning trajectories as a function of energy above the saddle.

References

[1] L. N. Virgin, T. C. Lyman, and R. B. Davis. Nonlinear dynamics of a ball rolling on a surface. *Am. J. Phys.*, 78(3):250–257, 2010.

[2] W. S. Koon, M. W. Lo, J. E. Marsden, and S. D. Ross. Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics. *Chaos*, 10:427–469, 2000.

[3] F. Gabern, W. S. Koon, J. E. Marsden, S. D. Ross, and T. Yanao. Application of tube dynamics to non-statistical reaction processes. *Few-Body Systems*, 38:167–172, 2006.

[4] R. S. MacKay. Flux over a saddle. *Physics Letters A*, 145:425–427, 1990.