Optimal capture trajectories using multiple gravity assists

Shane D. Ross

Engineering Science and Mechanics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24060, USA
sdross@vt.edu, http://www.esm.vt.edu

Stefan Jerg and Oliver Junge

Technische Universität München
Zentrum Mathematik
Boltzmannstr. 3
D-85747 Garching
{jerg,junge}@ma.tum.de, http://www-m3.ma.tum.de

Graph theoretic methods of optimal control in the presence of uncertainty are applied to a celestial mechanics problem. We find a fuel-efficient spacecraft trajectory which starts at infinity and is captured by the smaller member of a binary system, e.g., a moon of Jupiter, using multiple gravity assists.

Keywords: optimal control; three-body problem; celestial mechanics

1. Introduction

For low energy spacecraft trajectories such as multi-moon orbiters for the Jupiter system, multiple gravity assists by moons could be used in conjunction with ballistic capture to drastically decrease fuel usage. In this paper, we consider a spacecraft initially in a large orbit around Jupiter. Our goal is to use small impulsive controls to direct the spacecraft into a capture orbit about Callisto, the outermost icy moon of Jupiter. We consider the role of uncertainty, which is critical for space trajectories which are designed using chaotic dynamics. Our model is a family of symplectic twist maps which approximate the spacecraft's motion in the planar circular restricted three-body problem.[1] The maps capture well the dynamics of the full equations of motion; the phase space contains a connected chaotic zone where intersections between unstable resonant orbit manifolds provide the template for lanes of fast migration between orbits of different semimajor axes.

2. The Keplerian map

The example system we consider is the *Keplerian map*,[1]

$$\begin{pmatrix} \omega_{n+1} \\ K_{n+1} \end{pmatrix} = \begin{pmatrix} \omega_n - 2\pi(-2K_{n+1})^{-3/2} \pmod{2\pi} \\ K_n + \mu f(\omega_n; C_J, \bar{K}) \end{pmatrix} \tag{1}$$

of the cylinder $\mathcal{A} = S^1 \times \mathbb{R}$ onto itself. This two-dimensional symplectic twist map is an approximation of a Poincaré map of the planar restricted three-body problem, where the surface of section is at periapsis in the space of orbital elements. The map models a spacecraft on a near-Keplerian orbit about a central body of unit mass, where the spacecraft is perturbed by a smaller body of mass μ. The interaction of the spacecraft with the perturber is modeled as an impulsive kick at periapsis passage, encapsulated in the kick function f.

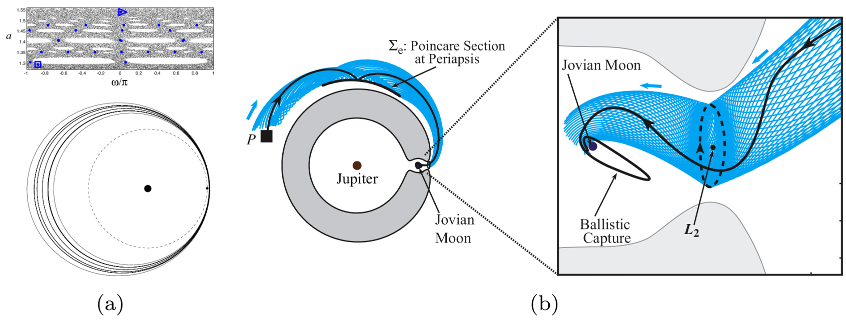

Fig. 1. (a) Upper panel: a phase space trajectory where the initial point is marked with a triangle and the final point with a square. Lower panel: the configuration space projections in an inertial frame for this trajectory. Jupiter and Callisto are shown at their initial positions, and Callisto's orbit is dashed. The uncontrolled spacecraft migration is from larger to smaller semimajor axes, keeping the periapsis direction roughly constant in inertial space. Both the spacecraft and Callisto orbit Jupiter in a counter-clockwise sense. The parameters used are $\mu = 5.667 \times 10^{-5}$, $C_J = 2.995$, $\bar{a} = -1/(2\bar{K}) = 1.35$, appropriate for a spacecraft in the Jupiter-Callisto system. (b) A spacecraft P inside a tube of gravitational capture orbits will find itself going from an orbit about Jupiter to an orbit about a moon. The spacecraft is initially inside a tube whose boundary is the stable invariant manifold of a periodic orbit about L_2. The three-dimensional tube, made up of individual trajectories, is shown as projected onto configuration space. The final intersection of the tube with Σ_e, a Poincaré map at periapsis in the exterior realm.

This map can be used for preliminary design of low energy trajectories which involve multiple gravity assists. A trajectory sent from Earth to the Jovian system, just grazing the orbit of the outermost icy moon Callisto, can migrate using little or no fuel from orbits with large apoapses to smaller ones. This is shown in Figure 1(a) in both the phase space and the inertial configuration space. From orbits slightly larger than Callisto's, the spacecraft can be captured into an orbit around the moon. The set of all capture orbits is a solid cylindrical tube in the phase space,[2,3] as shown in Figure 1(b). Followed backward in time this solid tube intersects transversally our Keplerian map, interpreted as a Poincaré surface-of-section. The resulting region is an *exit* from jovicentric orbits exterior to Callisto.

We can consider the location of an exit in the (ω, K)-plane as a target region for computing optimal capture trajectories. The details of the capture orbit around the moon are not considered here, but can be handled by other means at a finer scale.[4]

3. Control problem formulation

We are interested in studying the dynamics of the Keplerian map (1) subjected to control. We define a family of controlled Keplerian maps $F : \mathcal{A} \times U \to \mathcal{A}$

$$F\left(\begin{pmatrix}\omega_n\\ K_n\end{pmatrix}, u_n\right) = \begin{pmatrix}\omega_{n+1}\\ K_{n+1}\end{pmatrix} = \begin{pmatrix}\omega_n - 2\pi(-2K_{n+1})^{-3/2} \pmod{2\pi}\\ K_n + \mu f(\omega_n) + \alpha u_n\end{pmatrix} \quad (2)$$

where $u_n \in U = [-u_{\max}, u_{\max}]$, $u_{\max} \ll 1$, and the parametric dependence of f is understood. The term $\alpha = \alpha(C_J, \bar{K})$ is approximated as constant, where

$$\alpha = \sqrt{\frac{1}{\bar{a}}\left(\frac{1+\bar{e}}{1-\bar{e}}\right)}, \quad \text{with} \quad \bar{e} = \sqrt{1 - \left(\frac{C_J - \bar{a}}{2\bar{a}^{3/2}}\right)^2} \quad \text{and} \quad \bar{a} = -\frac{1}{2\bar{K}}. \quad (3)$$

Note that $F(\cdot, u_n)$ is area-preserving for any u_n. Physically, our control is modeled as a small impulsive thrust maneuver performed at periapsis n changing the speed by u_n. This increases K_n by an energy αu_n in addition to the natural dynamics term $\mu f(\omega_n)$.

Our goal is to control trajectories from a subset $S \subset \mathcal{A}$ to a target region $O \subset \mathcal{A}$. Additionally, we would like to minimize the total ΔV, while maintaining a reasonable transfer time. We model these requirements by considering the cost function $g : \mathcal{A} \times U \to [0, \infty)$,

$$g(a_n, u_n) = \frac{1}{2}|u_n|/u_{\max} + \frac{1}{2}\left(-\frac{1}{2K_n}\right)^{\frac{3}{2}},$$

where $a_n = (\omega_n, K_n)$ and our goal is to minimize the cost given by g that we accumulate along a controlled trajectory.

3.1. *Optimal feedback*

Standard methods for solving this (time discrete) optimal control problem include algorithms like *value* or *policy iteration*[5] which compute (approximations to) the *optimal value* function of the problem and a corresponding (approximate) optimal stabilizing *feedback* $u : \mathcal{A} \to U$. For a general *shortest path problem* on a continuous state space, as in our case, a more efficient technique has been proposed:[6–8]

For given $a \in \mathcal{A}$ and $\mathbf{u} \in U^{\mathbb{N}}$ there is a unique associated trajectory $(a_n(a, \mathbf{u}))_{\mathbf{n} \in \mathbb{N}}$ of (2). Let $\mathcal{U}(a) = \{\mathbf{u} \in U^{\mathbb{N}} : a_n(a, \mathbf{u}) \to O \text{ as } n \to \infty\}$ and $S = \{a \in \mathcal{A} : \mathcal{U}(a) \neq \emptyset\}$ the *stabilizable subset* $S \subset \mathcal{A}$. The total cost along a controlled trajectory is given by $J(a, \mathbf{u}) = \sum_{n=0}^{\infty} g(a_n(a, \mathbf{u}), u_n) \in [0, \infty]$.

The construction of the feedback is based on (an approximation to) the *optimal value function* $V : S \to [0, \infty]$, $V(x) = \inf_{\mathbf{u} \in \mathcal{U}(\mathbf{a})} J(a, \mathbf{u})$, which satisfies the *optimality principle*

$$V(a) = \inf_{u \in U} \{g(a, u) + V(F(a, u))\}. \tag{4}$$

The right hand side of this equation can be interpreted as an operator, acting on the function V, the *dynamic programming operator* L. If $\tilde{V}$ is an approximation to V, then one defines the feedback by

$$u(a) = \operatorname{argmin}_{u \in U} \left\{ g(a, u) + \tilde{V}(F(a, u)) \right\}, \tag{5}$$

whenever this minimum exists.

3.2. *Discretization*

We are going to approximate V by functions which are piecewise constant. Let $\mathcal{P}$ be a partition of $\mathcal{A}$, i.e. a collection of pairwise disjoint subsets which covers the state space $\mathcal{A}$. For a state $a \in \mathcal{A}$ we let $\rho(a)$ denote the element in the partition which contains a. Let $\mathbb{R}^{\mathcal{P}}$ be the subspace of the space $\mathbb{R}^{\mathcal{A}}$ of all real valued functions on $\mathcal{A}$ which are piecewise constant on the elements of the partition $\mathcal{P}$. The map $\varphi : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{P}}$, $\varphi[v](a) = \inf_{a' \in \rho(a)} v(a')$, is a projection onto $\mathbb{R}^{\mathcal{P}}$. We define the *discretized dynamic programming operator* $L_{\mathcal{P}} : \mathbb{R}^{\mathcal{P}} \to \mathbb{R}^{\mathcal{P}}$ by $L_{\mathcal{P}} = \varphi \circ L$. This operator has a unique fixed point $V_{\mathcal{P}}$ which satisfies $V_{\mathcal{P}}(O) = 0$ – the approximate (optimal) value function. One can show[8] that the fixed point equation $V_{\mathcal{P}} = L_{\mathcal{P}}[V_{\mathcal{P}}]$ is equivalent to the *discrete optimality principle*

$$V_{\mathcal{P}}(P) = \min_{P' \in \mathcal{F}(P)} \{\mathcal{G}(P, P') + V_{\mathcal{P}}(P')\},$$

where $V_{\mathcal{P}}(P) = V_{\mathcal{P}}(a)$ for any $a \in P \in \mathcal{P}$, the map $\mathcal{F}$ is given by

$$\mathcal{F}(P) = \{P' \in \mathcal{P} : P' \cap f(P, U) \neq \emptyset\} \tag{6}$$

and the cost function $\mathcal{G}$ by

$$\mathcal{G}(P, P') = \inf\{g(a, u) \mid a \in P, F(a, u) \in P', u \in U\}. \tag{7}$$

Note that the approximate value function $V_{\mathcal{P}}(P)$ is the length of the shortest path from P to $\rho(O)$ in the weighted directed graph $(\mathcal{P}, E)$, where the set of edges is defined by $E = \{(P, P') : P' \in \mathcal{F}(P)\}$ and the edge (P, P') is weighted by $\mathcal{G}(P, P')$. As such, it can be computed by, e.g., Dijkstra's algorithm.

In general, parameter uncertainties, modelling errors and small disturbances of the current state a_n may lead to a perturbed state $\tilde{a}_{n+1}$. Grüne and Junge[8] propose a generalization of the graph construction outlined above in order to cope with general disturbances. The following example computation is based on this general approach.

4. Low energy multiple gravity assists

We consider the Jupiter-Callisto system with state space $\mathcal{A} = [-\pi, \pi] \times [-0.4630, -0.03]$ which includes a start region corresponding to spacecraft initially in a large orbit around Jupiter. The target region O is the exit region leading to capture orbits around the moon. We use $u_{\max} = 5$ m/s (in normalized units). The computation of the value function is based on a partition of $\mathcal{A}$ into 2^{20} boxes of equal size (2^{10} boxes in each direction). We use 25 test points on an equidistant grid in each box in state space as well as 65 equally spaced points in the control range $[-u_{\max}, u_{\max}]$ in order to compute the graph 6, 7. Figure 2 shows the resulting approximate value function $\tilde{V}$ and a feedback trajectory starting from the initial point $a_0 = [\omega, K] = [0.036, -0.048]$ in the start region. The corresponding orbit in configuration space is also shown in Figure 2.

5. Conclusion

We applied a new feedback construction for discrete time optimal control problems with continuous state space which is based on graph theoretic methods to a celestial mechanics problem. We found a fuel-efficient spacecraft trajectory which starts in a large orbit around Jupiter and is captured by the smaller member of a binary system, e.g., a moon of Jupiter, using multiple gravity assists.

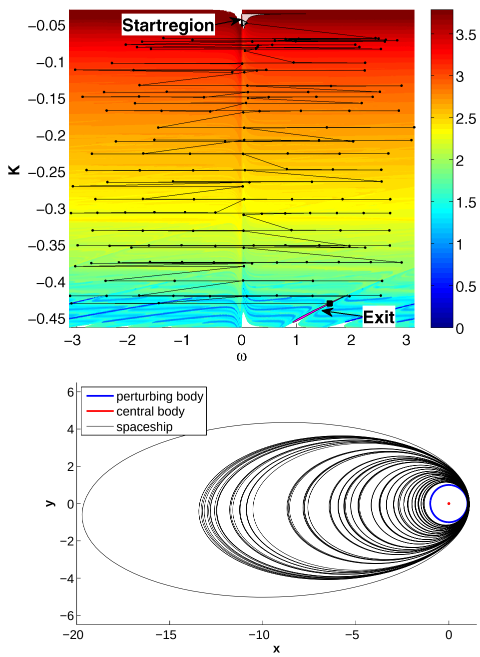

Fig. 2. *(Top)* The optimal value function and a feedback trajectory for the Keplerian map with $(\mu, C_J, \bar{a}) = (5.667 \times 10^{-5}, 2.995, 1.35)$. The initial point contained in the start region (gray) is marked by a triangle and the final point, which is contained in the exit region (magenta), by a square. *(Bottom)* projection onto configuration space of the controlled trajectory in an inertial frame (normalized units). The spacecraft migration is from larger to smaller semimajor axes, keeping the periapsis direction roughly constant in inertial space.

References

1. S. D. Ross and D. J. Scheeres, *SIAM Journal on Applied Dynamical Systems* **6**, 576 (2007).
2. W. S. Koon, M. W. Lo, J. E. Marsden and S. D. Ross, *Chaos* **10**, 427 (2000).
3. W. S. Koon, M. W. Lo, J. E. Marsden and S. D. Ross, Dynamical systems, the three-body problem and space mission design (2008), `http://www.shaneross.com/books`.
4. M. E. Paskowitz and D. J. Scheeres, *Journal of Guidance, Control, and Dynamics* **29**, 342 (2006).
5. D. P. Bertsekas, *Dynamic Programming and Optimal Control* (Athena Scientific, 1995).
6. O. Junge and H. M. Osinga, *ESAIM Control Optim. Calc. Var.* **10**, 259 (2004).
7. L. Grüne and O. Junge, *Systems Control Lett.* **54**, 169 (2005).
8. L. Grüne and O. Junge, *J. Optim. Theory Appl.* **136**, 411 (2008).