Chaos in Space and Time

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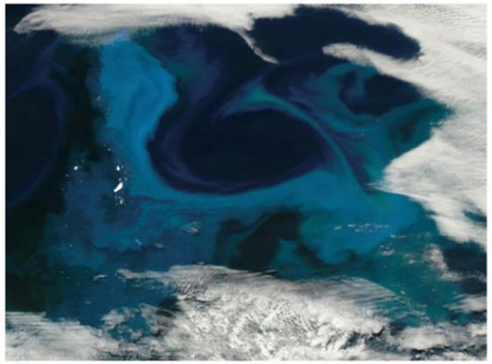
- Final Presentation
- ESM 6984 SS: Frontiers in Dynamical Systems

Why study chaos in space and time?

Many phenomena in nature are systems far-from equilibrium and exhibit chaotic dynamics in both space and time



NASA images



Falkowski Nature (2012)

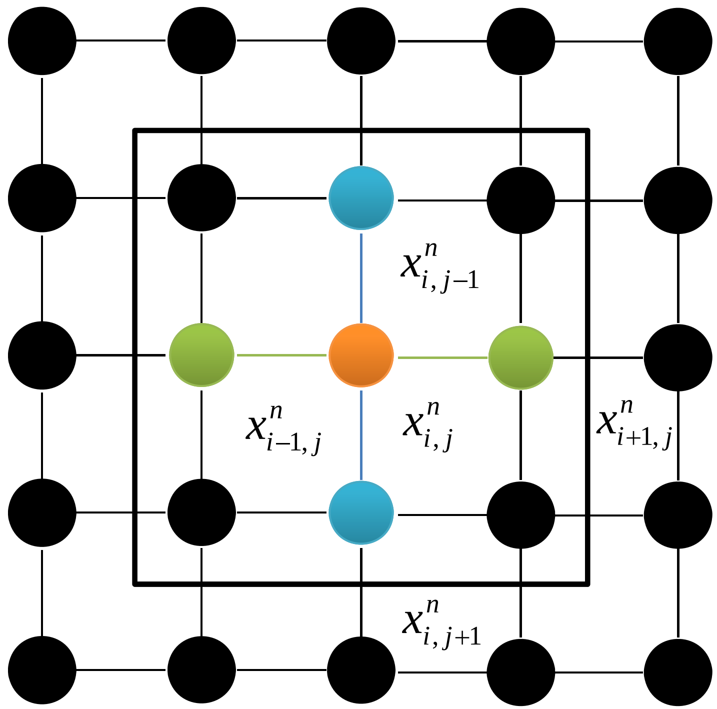
NASA image

Presentation Outline

- Lattice map with "diffusive" coupling
  - – Difference equations
- Calculating Lyapunov vectors
  - – System of ODEs
- Transport in complex flow
  - – Governing PDEs
- Conclusions
- Future directions

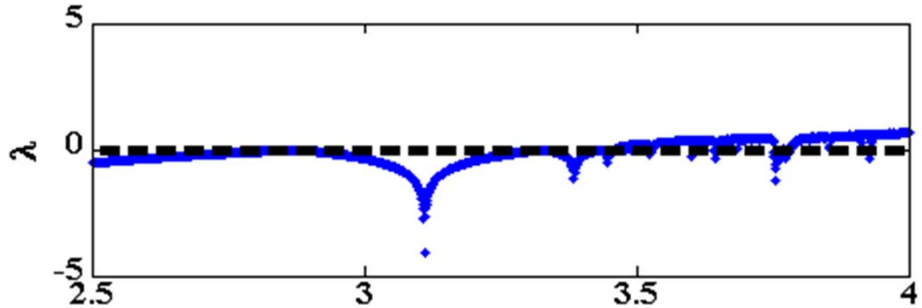
Coupled Map Lattice

$$x_{i,j}^{(n+1)} = f(x_{i,j}^n) + D[\frac{1}{2}(f(x_{i+1,j}^n) + f(x_{i-1}^n) + f(x_{i,j-1}^n) + f(x_{i,j+1}^n)) - f(x_{i,j}^n)]$$



$$\delta x_{i,j}^{(n+1)} = f'(x_{i,j}^n)\delta x_{i,j}^{(n)} + D[\frac{1}{2}(f'(x_{i+1,j}^n)\delta x_{i+1,j}^{(n)} + f'(x_{i-1,j}^n)\delta x_{i-1,j}^{(n)} + f'(x_{i,j+1}^n)\delta x_{i,j+1}^{(n)} + f'(x_{i,j-1}^n)\delta x_{i,j-1}^{(n)}) - f'(x_{i,j}^n)\delta x_{i,j}^{(n)}]$$

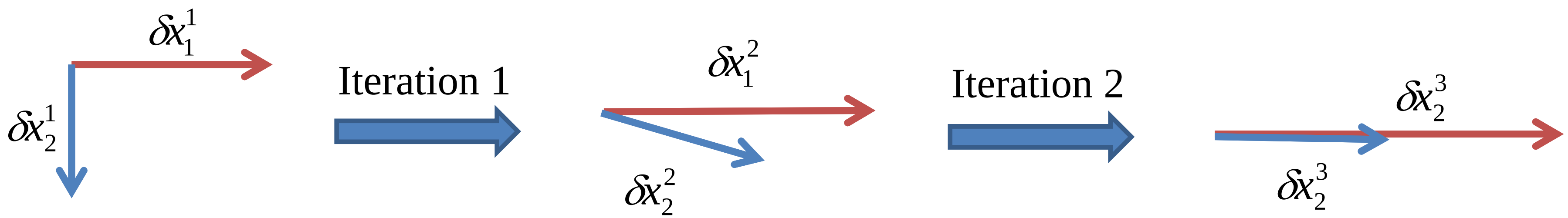
$$f(x_i^n) = a x_i^n (1 - x_i^n)$$



Coupled Map Lattice

Gram-Schmidt Method

In a chaotic system , each vector tends to fall along the local direction of most rapid growth



To overcome the problem, Gram-Schmidt method was implemented

$$\delta\vec{x}_1{}'(n) = \frac{\delta\vec{x}_1(n)}{\left\|\delta\vec{x}_1(n)\right\|},$$

$$\delta\vec{x}_2{}'(n) = \frac{\delta\vec{x}_2(n) - \left\langle \delta\vec{x}_2(n), \delta\vec{x}_1{}'(n)\right\rangle \delta\vec{x}_1'(n)}{\left\|\delta\vec{x}_2(n) - \left\langle \delta\vec{x}_2(n), \delta\vec{x}_1{}'(n)\right\rangle \delta\vec{x}_1'(n)\right\|},$$

$$\vdots$$

$$\delta\vec{x}_n{}'(n) = \frac{\delta\vec{x}_n(n) - \left\langle \delta\vec{x}_n(n), \delta\vec{x}_{n-1}{}'(n)\right\rangle \delta\vec{x}_{n-1}'(n) - \cdots - \left\langle \delta\vec{x}_n(n), \delta\vec{x}_1{}'(n)\right\rangle \delta\vec{x}_1'(n)}{\left\|\delta\vec{x}_n(n) - \left\langle \delta\vec{x}_n(n), \delta\vec{x}_{n-1}{}'(n)\right\rangle \delta\vec{x}_{n-1}'(n) - \cdots - \left\langle \delta\vec{x}_n(n), \delta\vec{x}_1{}'(n)\right\rangle \delta\vec{x}_1'(n)\right\|}.$$

$$\lambda_1^* = \lim_{n\to\infty} \frac{1}{n\Delta t} \sum_{i=1}^{n} \ln\left\|\delta\vec{x}_1(n)\right\|,$$

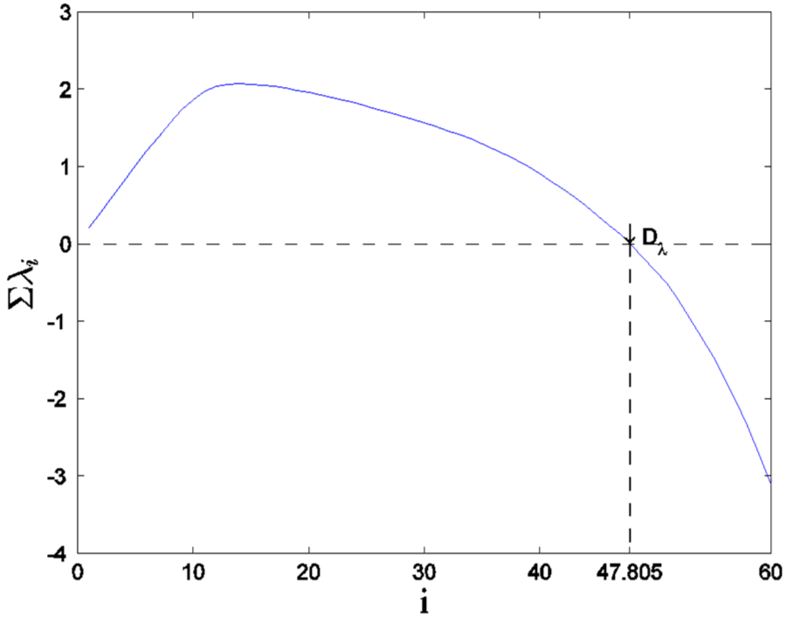
$$\lambda_2^* = \lim_{n\to\infty} \frac{1}{n\Delta t} \sum_{i=1}^{n} \ln\left\|\delta\vec{x}_2^*\right\|,$$

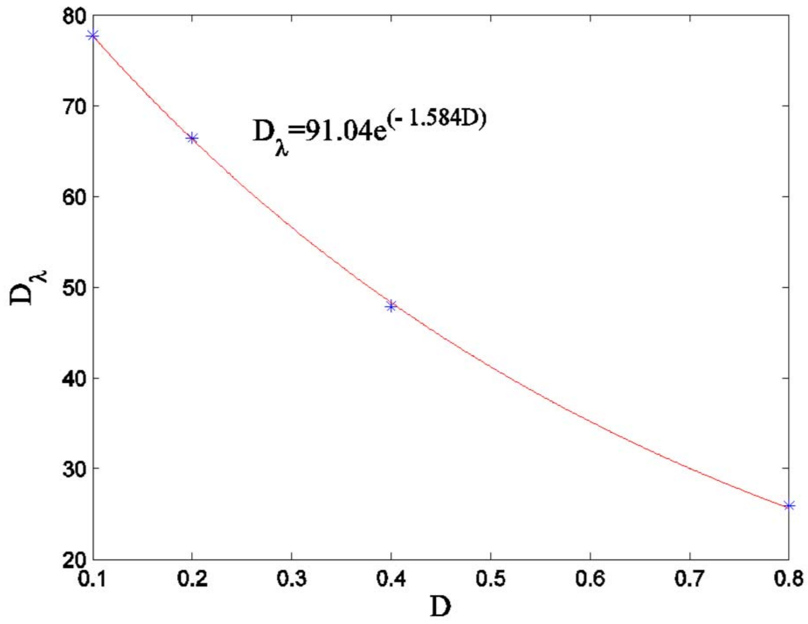
$$\vdots$$

$$\lambda_n^* = \lim_{n\to\infty} \frac{1}{n\Delta t} \sum_{i=1}^{n} \ln\left\|\delta\vec{x}_n^*\right\|.$$

Coupled Map Lattice

One-dimensional

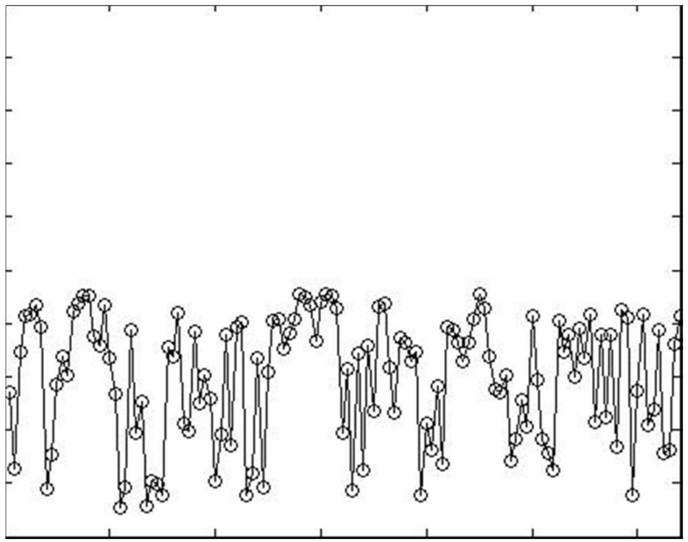




$$D_\lambda = j + \frac{\sum_{i=1}^{j} \lambda_i}{\left|\lambda_{j+1}\right|}.$$

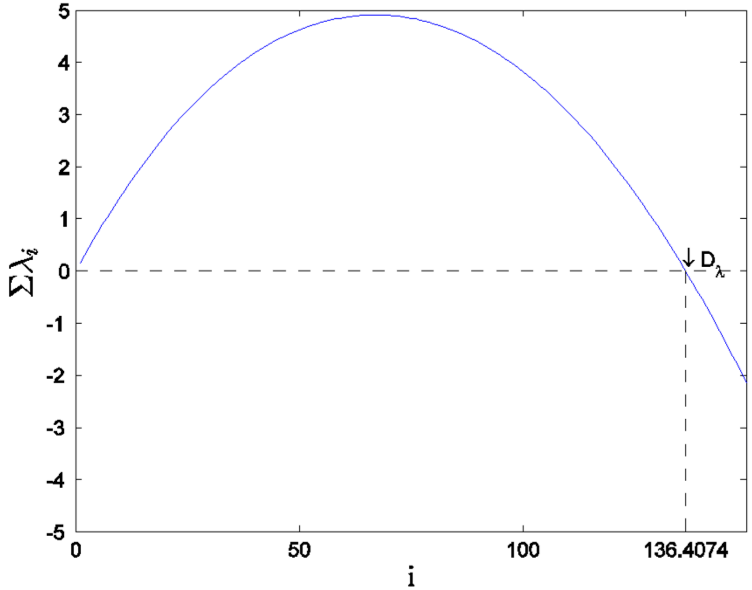
$a = 3.7$

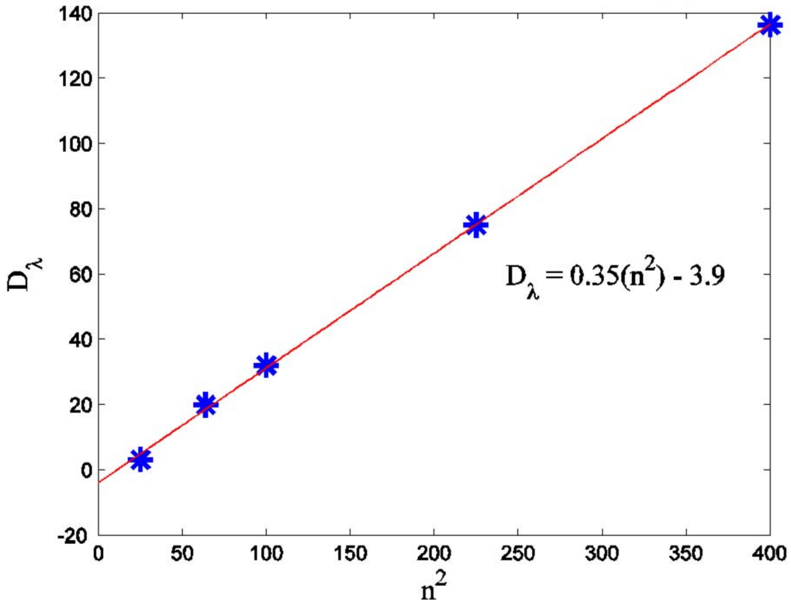
$D = 0.4$

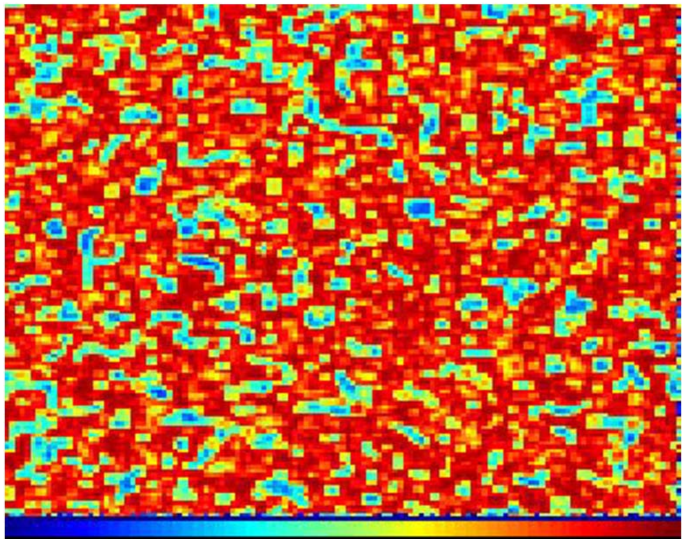


Coupled Map Lattice

Two-dimensional







$$a = 3.7$$
$$D = 0.4$$

$$D_\lambda \propto \Gamma^d$$

Lyapunov Vectors

Covariant Lyapunov Vectors

Pros:

- True direction in phase space.
- Reflect the direction of perturbation
- Test hyperbolicity

Cons:

- Difficult to calculate
- Algorithm only recently available(Ginelli (2007) and Pazo (2007))

Orthogonal Lyapunov Vectors

Pros:

- Easy to calculate
- Leading order Lyapunov vector is in correct direction
- Can calculate fractal dimension

Cons

- Lose all direction except leading order

Lorenz System

$$\frac{dx}{dt} = \sigma(x - y)$$

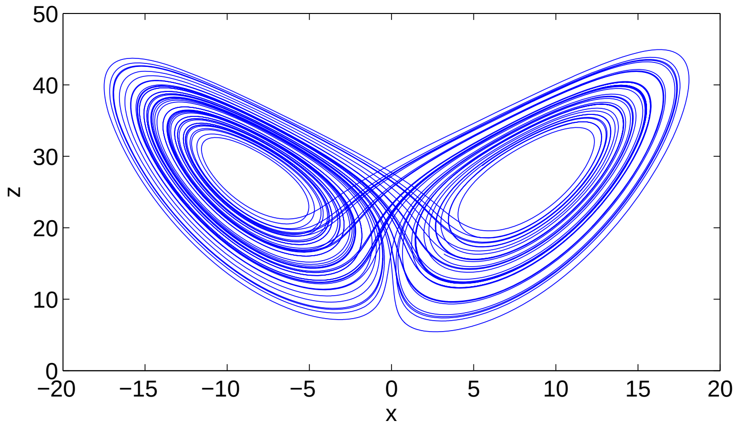
$$\frac{dy}{dt} = x(\rho - z) - y$$

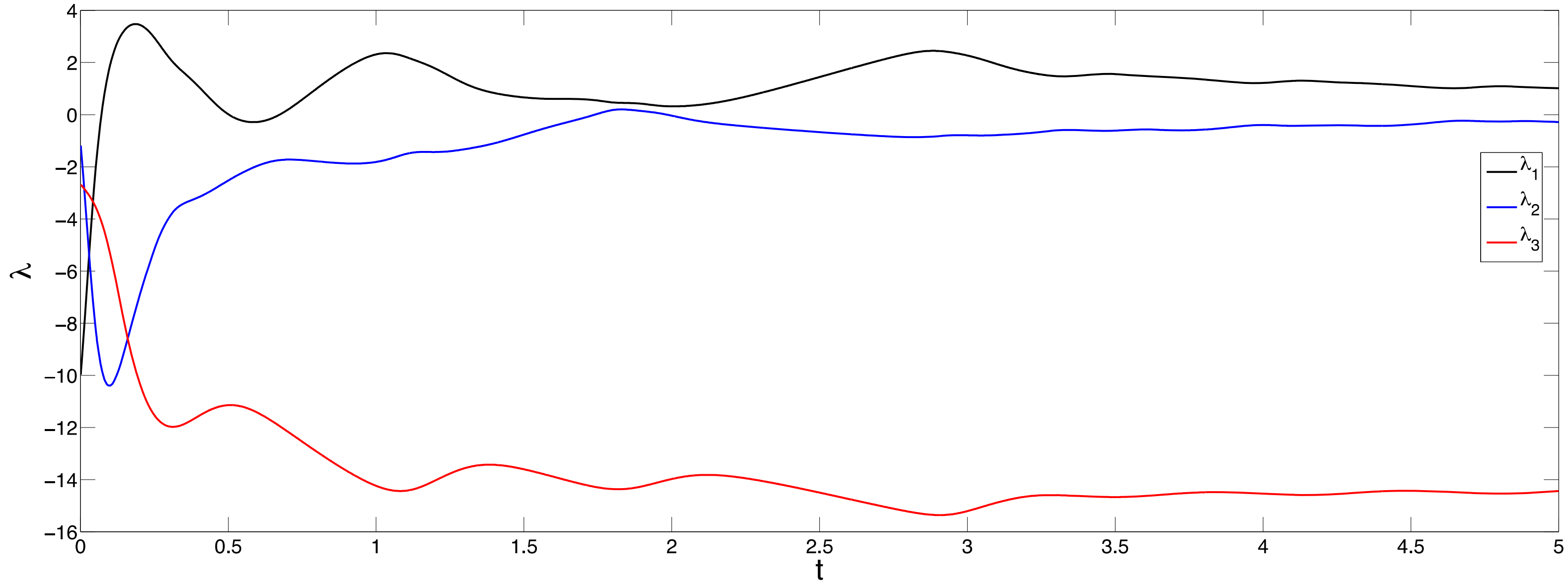
$$\frac{dz}{dt} = xy - \beta z$$

$$\sigma = 10$$

$$\rho = 28$$

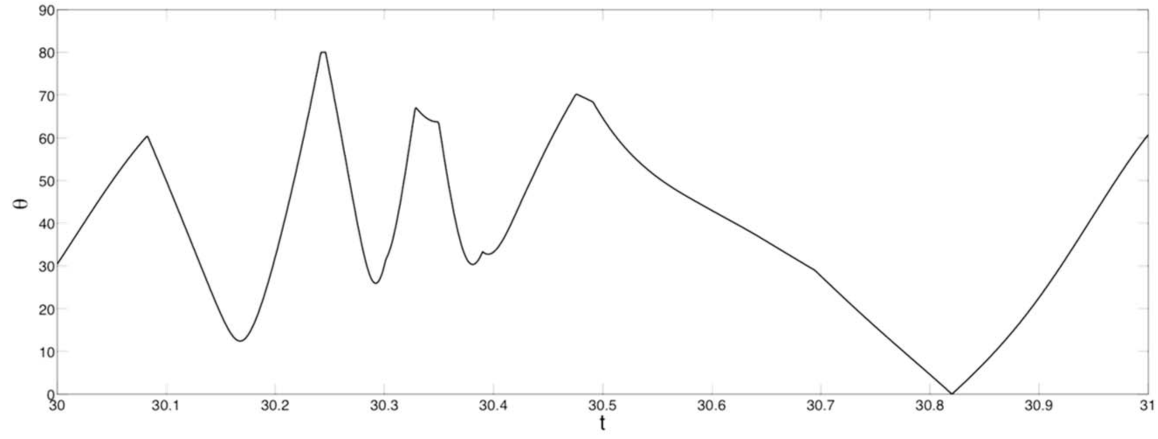
$$\beta = \frac{8}{3}$$

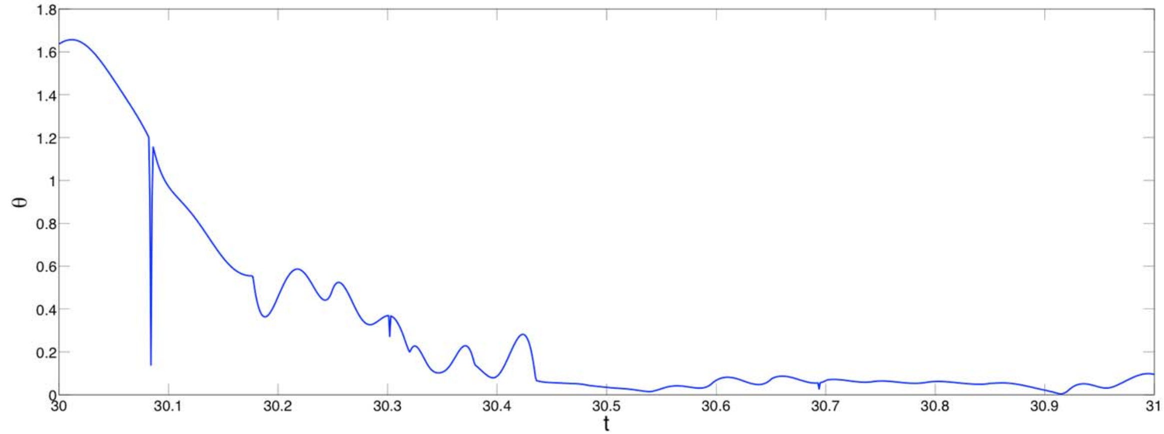




Results of Covariant Lyapunov Vectors

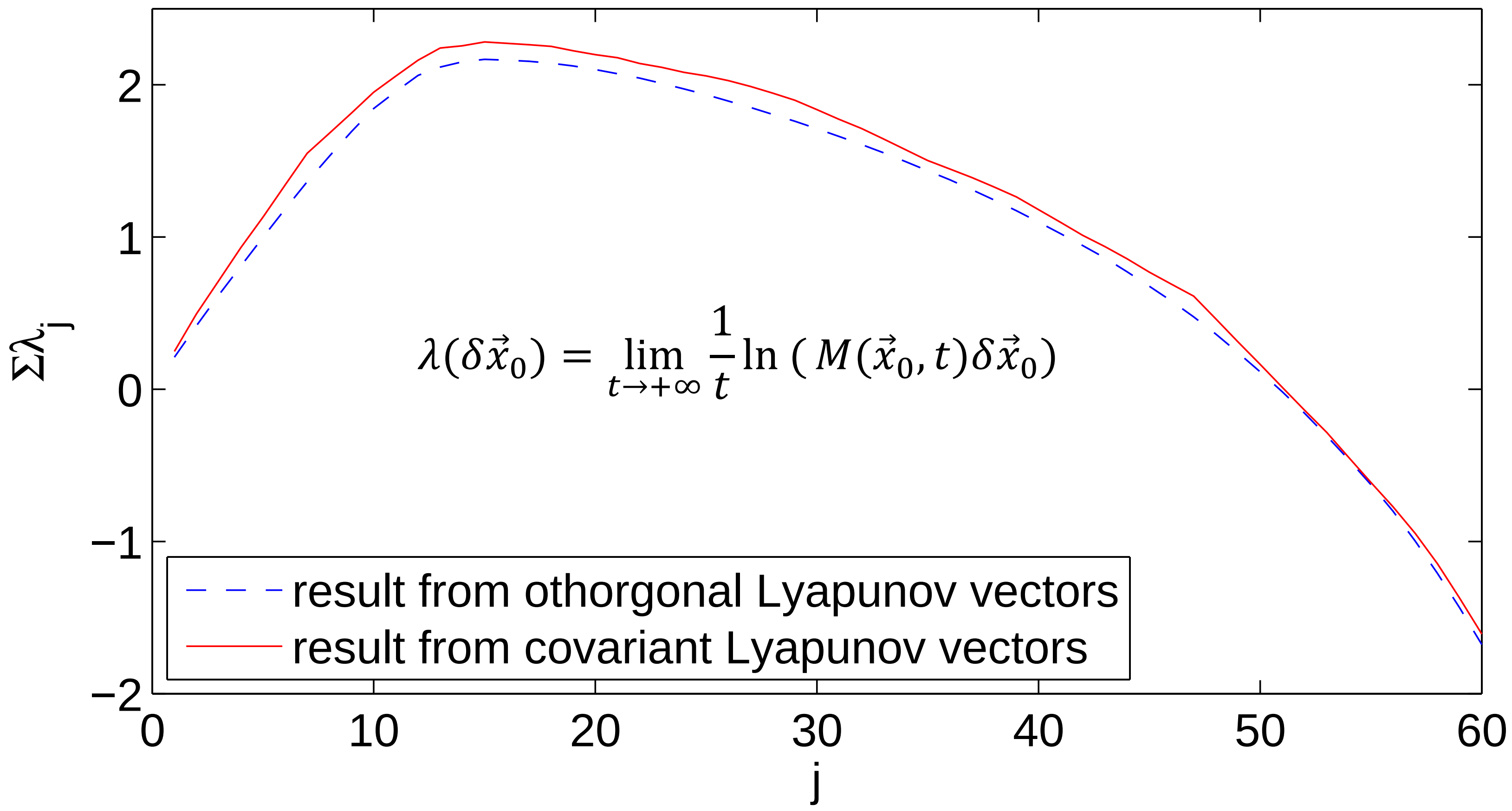
The direction of the second covariant Lyapunov vector and the direction of the tangent vector should be same.





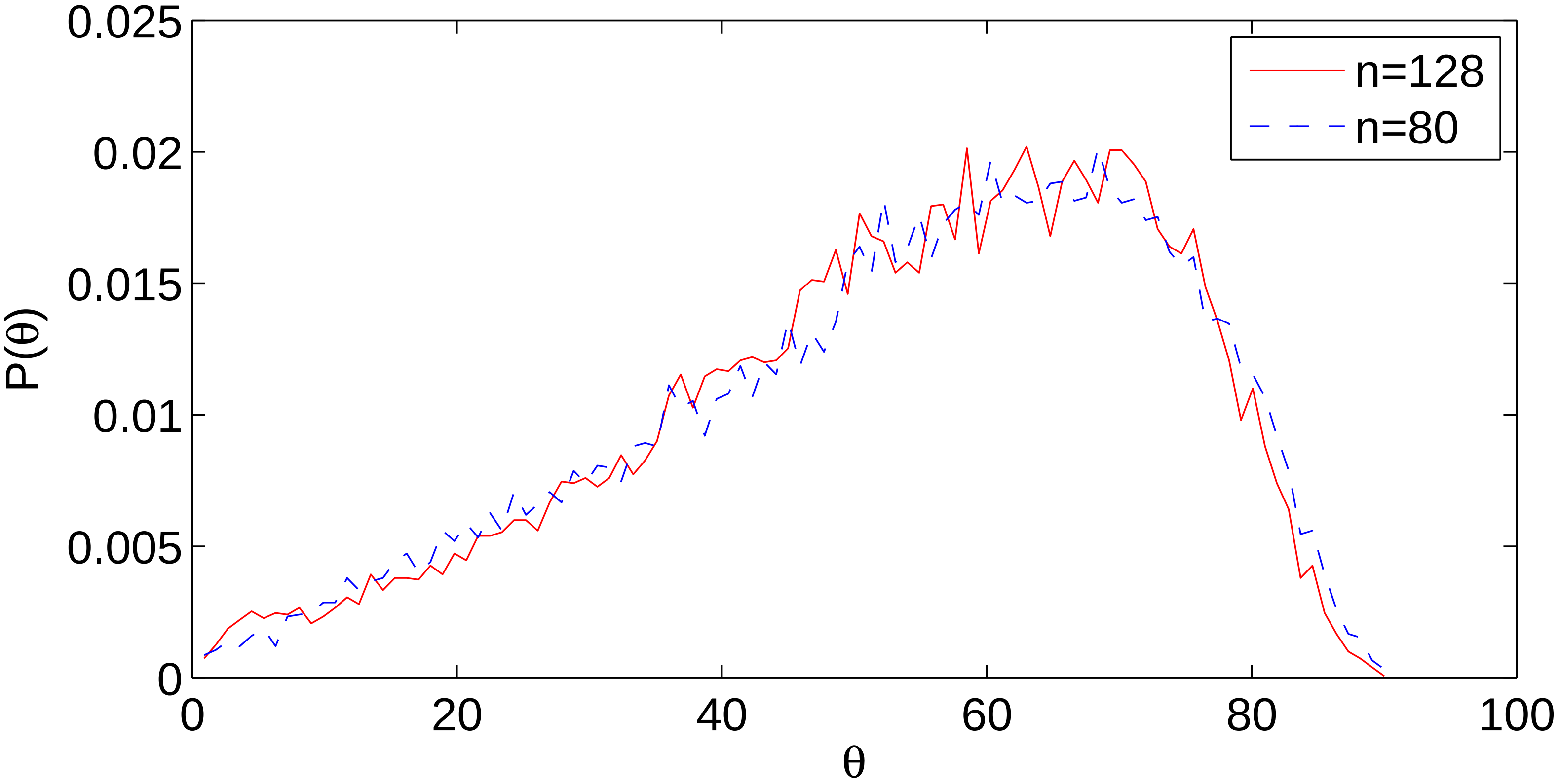
Results of Covariant Lyapunov Vectors in Coupled Map Lattice 1D

The Lyapunov exponents from different algorithm should agree with each other.



Hyperbolicity in Coupled Map Lattice 1D

The size of system does not influence the hyperbolicity of the system.



Transport in Complex Flows

Boussinesq Equations

$$\sigma^{-1}\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right) = -\vec{\nabla}p + \vec{\nabla}^2\vec{u} + RT\hat{z}$$

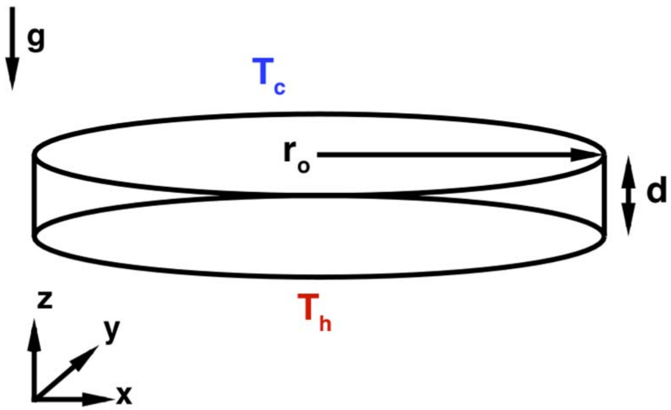
$$\left(\frac{\partial T}{\partial t} + (\vec{u}\cdot\vec{\nabla})T\right) = \vec{\nabla}^2 T$$

$$\vec{\nabla}\cdot\vec{u} = 0$$

Advection-Diffusion Equation

$$\frac{\partial c}{\partial t} + (\vec{u}\cdot\vec{\nabla})c = L\vec{\nabla}^2 c$$

$$R = \frac{\alpha g d^3}{\nu\kappa}\Delta T \qquad L = \frac{D}{\kappa}$$

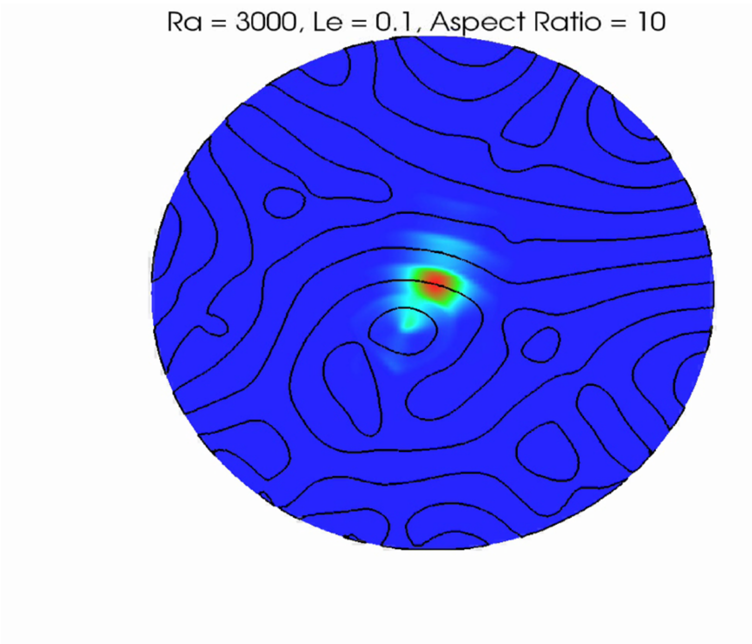


(a)

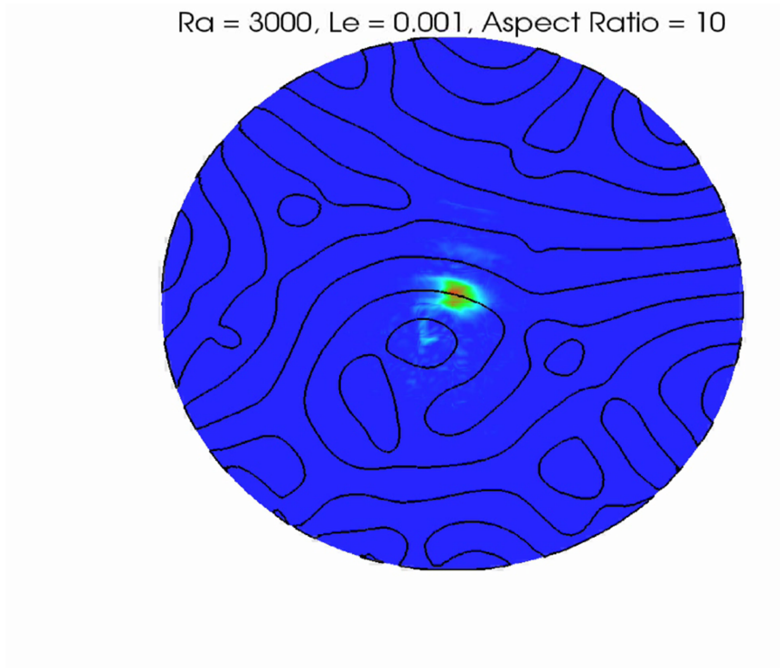
Ning et al. (2009)

Direct Numerical Simulations

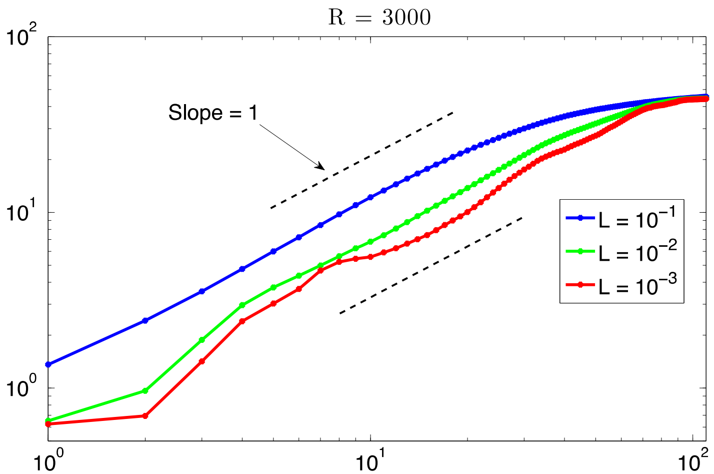
Pr= 1

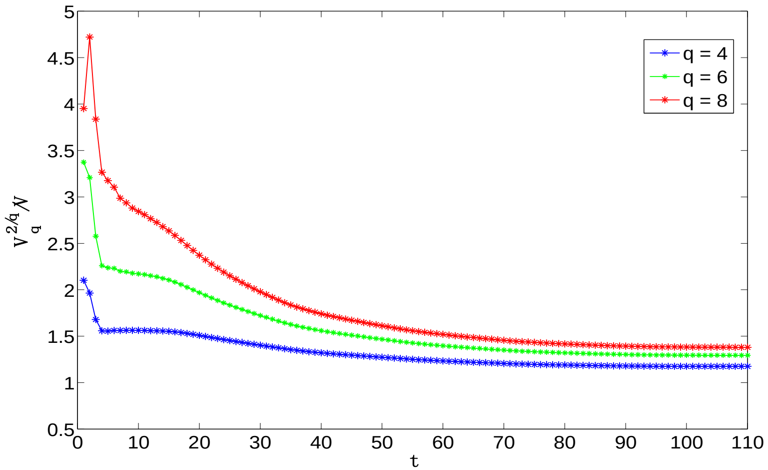


Pr= 1



Spreading of Species





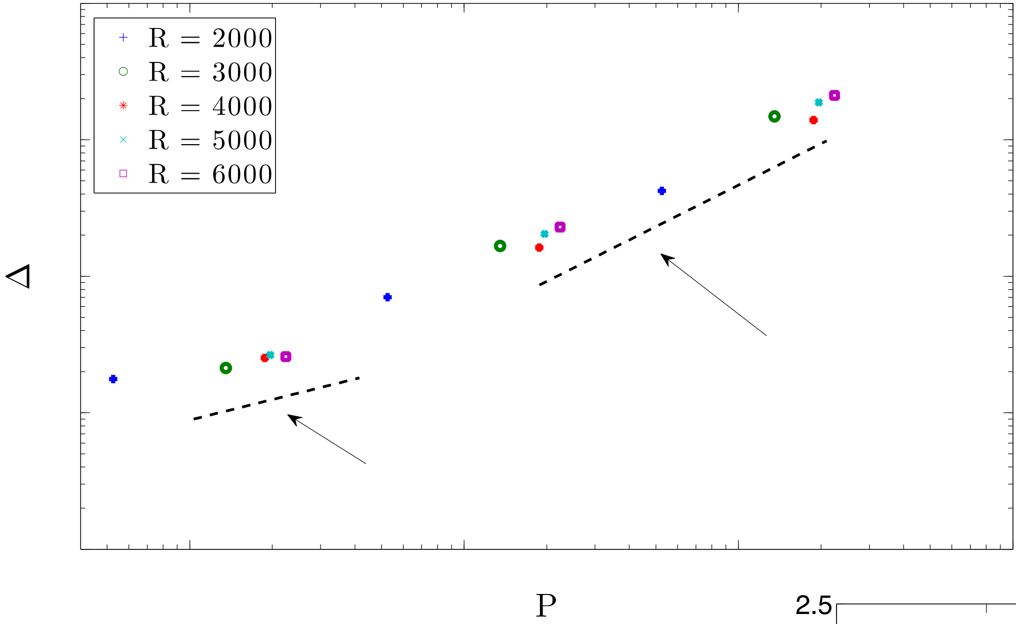
$$[L]^2 \propto D[T]$$

$$\frac{V_q^{2/q}}{V} = const.$$

Normal Diffusive Transport

$$\frac{\partial \tilde{c}}{\partial t} = L^* \frac{\partial^2 \tilde{c}}{\partial r^2}$$

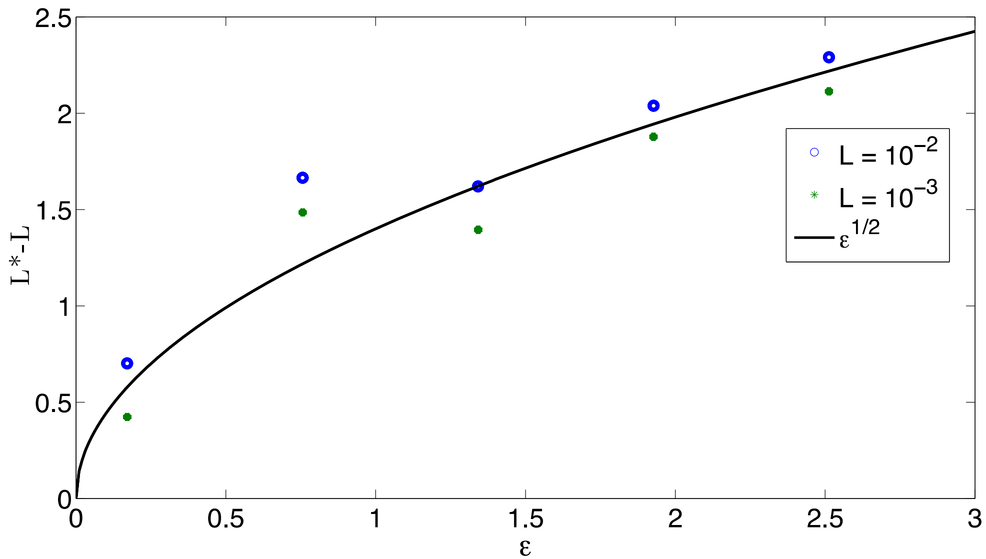
Enhanced Transport



$$\Delta = \frac{L^* - L}{L}$$

$$P = \frac{\left\|\vec{u}\right\|}{L}$$

$$L^* - L \propto \left(\frac{R - R_c}{R_c}\right)^{1/2}$$



Conclusions and Future Directions

- Fractal dimension proportional to map lattice size
- Hyperbolicity was not influenced by lattice size
- Two transport enhancement regimes due to spatiotemporal chaotic flow field
- Calculate covariant Lyapunov vectors in Rayleigh-Bénard convection
- Conduct formal study on influence of system size