Dynamical Systems and Space Mission Design

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Halo Orbit and Its Computation: Outline

- In Lecture 5A, we have covered
  - Importance of halo orbits.
  - Finding periodic solutions of the linearized equations.
  - Highlights on 3rd order approximation of a halo orbit.
  - Using a textbook example to illustrate Lindstedt-Poincaré method.
- In Lecture 5B, we will cover
  - Use L.P. method to find a 3rd order approximation of a halo orbit.
  - Finding a halo orbit numerically via differential correction.
  - Orbit structure near L_1 and L_2.

Review of Lindstedt-Poincaré Method

- To avoid **secure** terms, Lindstedt-Poincaré method
  - Notices **non-linearity** alters **frequency** λ to $\lambda\omega(\epsilon)$.
  - Introduce new independent variable $\tau = \omega(\epsilon)t$:

$$t = \tau\omega^{-1} = \tau(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots).$$

  - Rewrite equation using τ as independent variable:

$$q'' + (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^2(q + \epsilon q^3) = 0.$$

  - Expand periodic solution in a power series of ϵ:

$$q = \sum_{n=0}^{\infty} \epsilon^n q_n(\tau) = q_0(\tau) + \epsilon q_1(\tau) + \epsilon^2 q_2(\tau) + \cdots$$

- By substituting q into equation and equating terms in ϵ^n:

$$\begin{aligned} q_0'' + q_0 &= 0, \\ q_1'' + q_1 &= -q_0^3 - 2\omega_1 q_0, \\ q_2'' + q_2 &= -3q_0^2 q_1 - 2\omega_1(q_1 + q_0^3) + (\omega_1^2 + 2\omega_2)q_0, \end{aligned}$$

Review of Lindstedt-Poincaré Method

- Remove **secular** terms by choosing suitable ω_n.
  - Solution of 1st equation: $q_0 = a\cos(\tau + \tau_0)$.
  - Substitute $q_0 = a\cos(\tau + \tau_0)$ into 2nd equation

$$\begin{aligned} q_1'' + q_1 &= -a^3 \cos^3(\tau + \tau_0) - 2\omega_1 a \cos(\tau + \tau_0) \\ &= -\frac{1}{4}a^3 \cos 3(\tau + \tau_0) - (\frac{3}{4}a^2 + 2\omega_1) a\cos(\tau + \tau_0). \end{aligned}$$

  - Set $\omega_1 = -3a^2/8$ to remove $\cos(\tau + \tau_0)$ and **secular** term.
- Therefore, to 1st order of ϵ, we have **periodic** solution

$$q = a\cos(\omega t + \tau_0) + \frac{1}{32}\epsilon \cos 3(\omega t + \tau_0) + o(\epsilon^2).$$

with

$$\omega = (1 - \frac{3}{8}\epsilon a^2 - \frac{15}{256}\epsilon^2 a^4 + o(\epsilon^3).$$

- **Lindstedt-Poincaré method consists in successive adjustments of frequencies.**

Lindstedt Poincaré Method: Nonlinear Expansion

- CR3BP equations can be developed using Legendre polynomial P_n

$$\ddot{x} - 2\dot{y} - (1 + 2c_2)x = \frac{\partial}{\partial x}\sum_{n\geq 3} c_n \rho^n P_n(\frac{x}{\rho})$$
$$\ddot{y} + 2\dot{x} + (c_2 - 1)y = \frac{\partial}{\partial y}\sum_{n\geq 3} c_n \rho^n P_n(\frac{x}{\rho})$$
$$\ddot{z} + c_2 z = \frac{\partial}{\partial z}\sum_{n\geq 3} c_n \rho^n P_n(\frac{x}{\rho})$$

where $\rho^2 = x^2 + y^2 + z^2$, and $c_n = \gamma^{-3}(\mu + (-1)^n(1-\mu)(\frac{\gamma}{1-\gamma})^{n+1})$.

- Useful if successive approximation solution procedure is carried to high order via algebraic manipulation software programs.

$$P_n(\frac{x}{\rho}) = \frac{x}{\rho}(\frac{2n-1}{n})P_{n-1}(\frac{x}{\rho}) - (\frac{n-1}{n})P_{n-2}(\frac{x}{\rho}).$$

- Recall that $\rho < 1$.

Lindstedt Poincaré Method: 3rd Order Expansion

- 3rd order approximation used in Richardson [1980]:

$$
\begin{aligned}
\ddot{x} - 2\dot{y} - (1 + 2c_2)x &= \frac{3}{2}c_3(2x^2 - y^2 - z^2) \\
&\quad + 2c_4 x(2x^2 - 3y^2 - 3z^2) + o(4), \\
\ddot{y} + 2\dot{x} + (c_2 - 1)y &= -3c_3 xy - \frac{3}{2}c_4 y(4x^2 - y^2 - z^2) + o(4), \\
\ddot{z} + c_2 z &= -3c_3 xz - \frac{3}{2}c_4 z(4x^2 - y^2 - z^2) + o(4).
\end{aligned}
$$

Construction of Periodic Solutions

- Recall that solution to the linearized equations

$$\begin{aligned} \ddot{x} - 2\dot{y} - (1 + 2c_2)x &= 0 \\ \ddot{y} + 2\dot{x} + (c_2 - 1)y &= 0 \\ \ddot{z} + c_2 z &= 0 \end{aligned}$$

has the following form

$$\begin{aligned} x &= -A_x \cos(\lambda t + \phi) \\ y &= kA_x \sin(\lambda t + \phi) \\ z &= A_z \sin(\nu t + \psi) \end{aligned}$$

- Halo orbits are obtained if amplitudes A_x and A_z of linearized solution are large enoug so that nonlinear contributions makes eigen-frequencies equal ($\lambda = \nu$).
- This linearized solution ($\lambda = \nu$) is the seed for constructing successve approximations.

Construction of Periodic Solutions

- We would like to rewrite linearized equations in following form:

$$\begin{aligned} \ddot{x} - 2\dot{y} - (1 + 2c_2)x &= 0 \\ \ddot{y} + 2\dot{x} + (c_2 - 1)y &= 0 \\ \ddot{z} + \lambda^2 z &= 0 \end{aligned}$$

  which has a periodic solution with frequency λ.

- Need to have a correction term $\Delta = \lambda^2 - c_2$ for high order approximations.

$$\ddot{z} + \lambda^2 z = -3c_3 xz - \frac{3}{2}c_4 z(4x^2 - y^2 - z^2) + \Delta z + o(4).$$

Lindstedt-Poincaré Method

- Richardson [1980] developed a 3rd order periodic solution using a L.P. type successive approximations.
  - To remove **secular** terms, a new independent variable τ and a frequency connection ω are introduced via

$$\tau = \omega t.$$

  - Here,

$$\omega = 1 + \sum_{n \geq 1} \omega_n, \qquad \omega_n < 1.$$

  - The ω_n are assumed to be o(A_z^n) and are chosen to remove **secure** terms.
  - Notice that $A_z << 1$ in normalized unit and it plays the role of ϵ.

Lindstedt-Poincaré Method

- Equations are then written in terms of new independent variable τ

$$
\begin{aligned}
\omega^2 x'' - 2\omega y' - (1+2c_2)x &= \frac{3}{2}c_3(2x^2 - y^2 - z^2) \\
&\quad + 2c_4 x(2x^2 - 3y^2 - 3z^2) + o(4), \\
\omega^2 y'' + 2\omega x' + (c_2 - 1)y &= -3c_3 xy \\
&\quad - \frac{3}{2}c_4 y(4x^2 - y^2 - z^2) + o(4), \\
\omega^2 z'' + \lambda^2 z &= -3c_3 xz \\
&\quad - \frac{3}{2}c_4 z(4x^2 - y^2 - z^2) + \Delta z + o(4).
\end{aligned}
$$

- 3rd order successive approximation solution is a lengthy process. Here are some highlights:
  - Generating solution is linearized solution with t replaced by τ

$$
\begin{aligned}
x &= -A_x \cos(\lambda\tau + \phi) \\
y &= kA_x \sin(\lambda\tau + \phi) \\
z &= A_z \sin(\lambda\tau + \psi)
\end{aligned}
$$

Lindstedt-Poincaré Method

- Some highlights:
  - Look for general solutions of the following type:

$$x = \sum_{n \geq 0} a_n \cos n\tau_1, \quad y = \sum_{n \geq 0} b_n \sin n\tau_1, \quad z = \sum_{n \geq 0} c_n \cos n\tau_1,$$

    where $\tau_1 = \lambda\tau + \phi = \lambda\omega t + \phi$.
  - It is found that

$$\omega_1 = 0, \qquad \omega_2 = s_1 A_x^2 + s_2 A_z^2,$$

    which give the frequence $\lambda\omega$ ($\omega = 1 + \omega_1 + \omega_2 + \cdots$) and the period T ($T = 2\pi/\lambda\omega$) of a halo orbit.
  - To remove all **secular** terms, it is also necessary to specify **amplitude** and **phase-angle** constraint relationships:

$$\begin{aligned} l_1 A_x^2 + l_2 A_z^2 + \Delta &= 0, \\ \psi - \phi &= m\pi/2, \qquad m = 1, 3. \end{aligned}$$

Halo Orbits in 3rd Order Approximation

- 3rd order solution in Richardson [1980]:

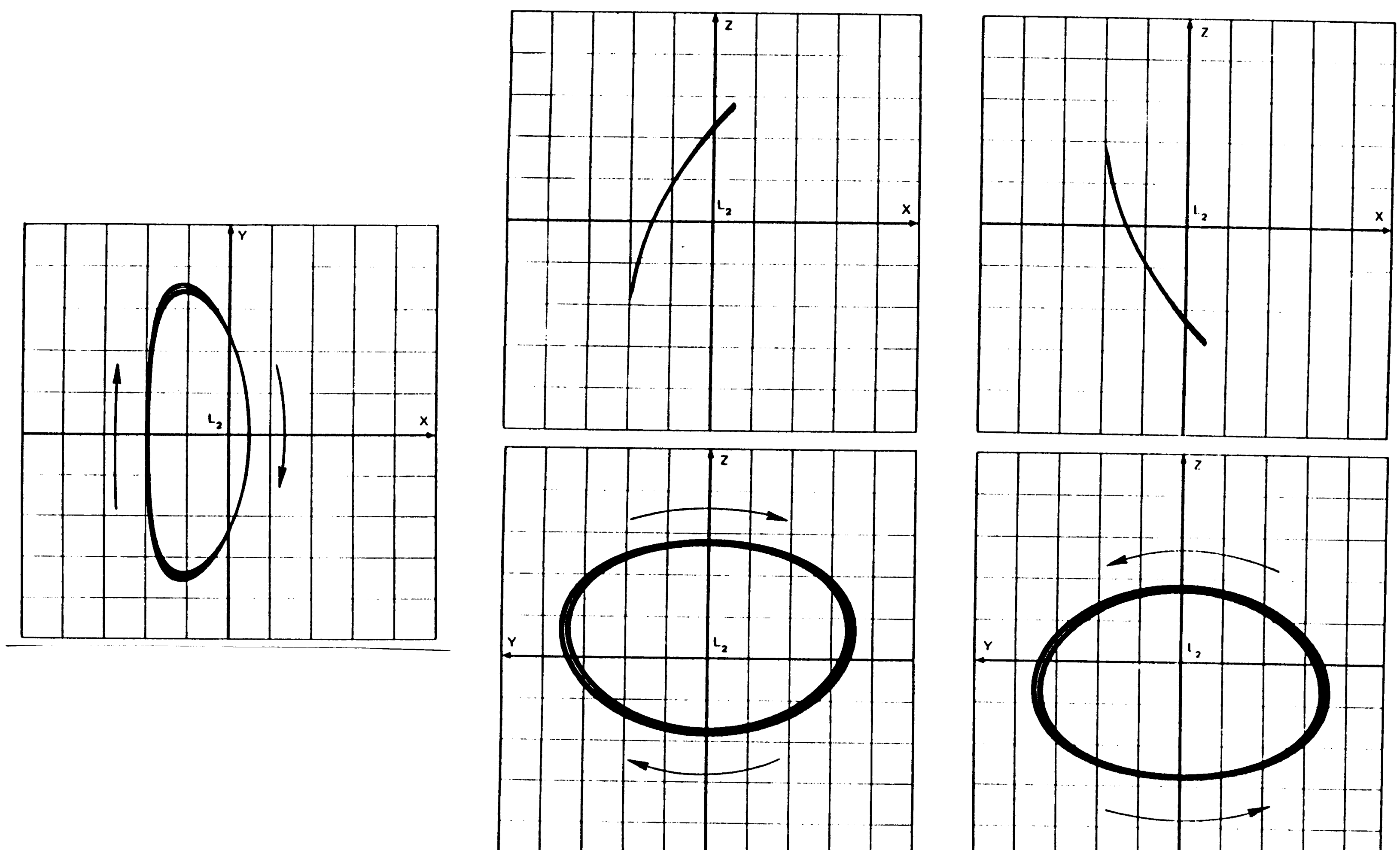
$$
\begin{aligned}
x &= a_{21}A_x^2 + a_{22}A_z^2 - A_x \cos\tau_1 \\
&\quad + (a_{23}A_x^2 - a_{24}A_z^2)\cos 2\tau_1 + (a_{31}A_x^3 - a_{32}A_x A_z^2)\cos 3\tau_1, \\
y &= kA_x \sin\tau_1 \\
&\quad + (b_{21}A_x^2 - b_{22}A_z^2)\sin 2\tau_1 + (b_{31}A_x^3 - b_{32}A_x A_z^2)\sin 3\tau_1, \\
z &= \delta_m A_z \cos\tau_1 \\
&\quad + \delta_m d_{21}A_x A_z(\cos 2\tau_1 - 3) + \delta_m(d_{32}A_z A_x^2 - d_{31}A_z^3)\cos 3\tau_1.
\end{aligned}
$$

where $\tau_1 = \lambda\tau + \phi$ and $\delta_m = 2 - m, m = 1, 3$.

- 2 solution branches are obtained according to whether $m = 1$ or $m = 3$.

Halo Orbit Phase-angle Relationship

- Bifurcation manifests through phase-angle relationship:
  - For $m = 1$, $A_z > 0$. Northern halo.
  - For $m = 3$, $A_z < 0$. Southern halo.
  - Northern & southern halos are mirror images across xy-plane.



Halo Orbit Amplitude Constraint Relationship

- For halo orbits, we have amplitude constraint relationship

$$l_1 A_x^2 + l_2 A_z^2 + \Delta = 0.$$

  - Minimim value for A_x to have a halo orbit $(A_z > 0)$ is $\sqrt{|\Delta / l_1|}$, which is about $200,000$ km.
  - Halo orbit can be characterized completely by A_z.

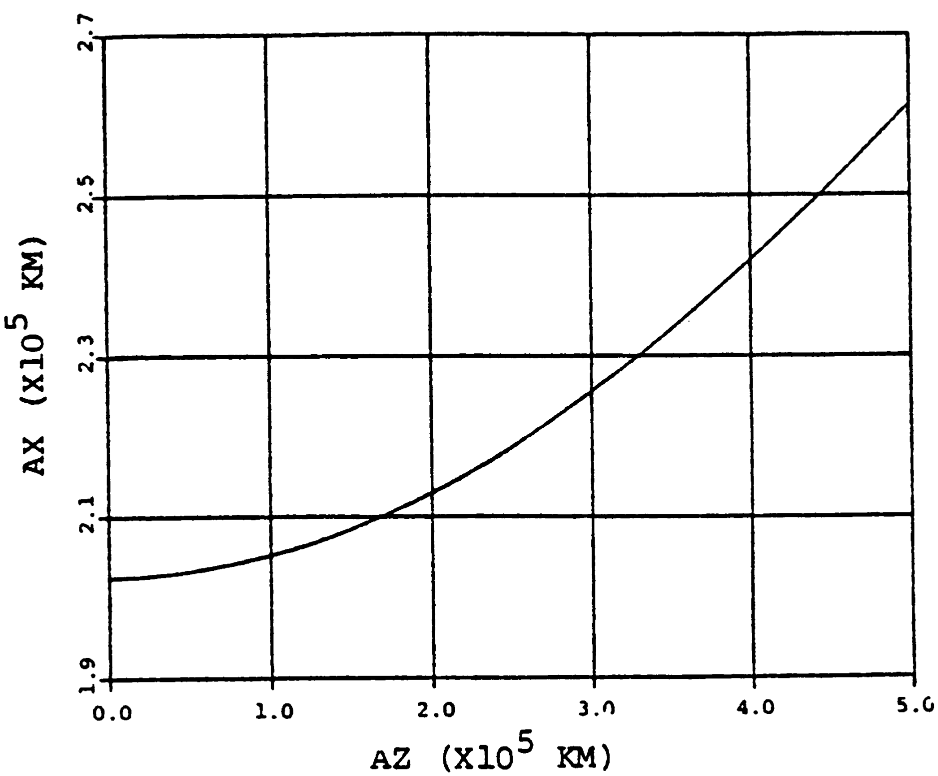


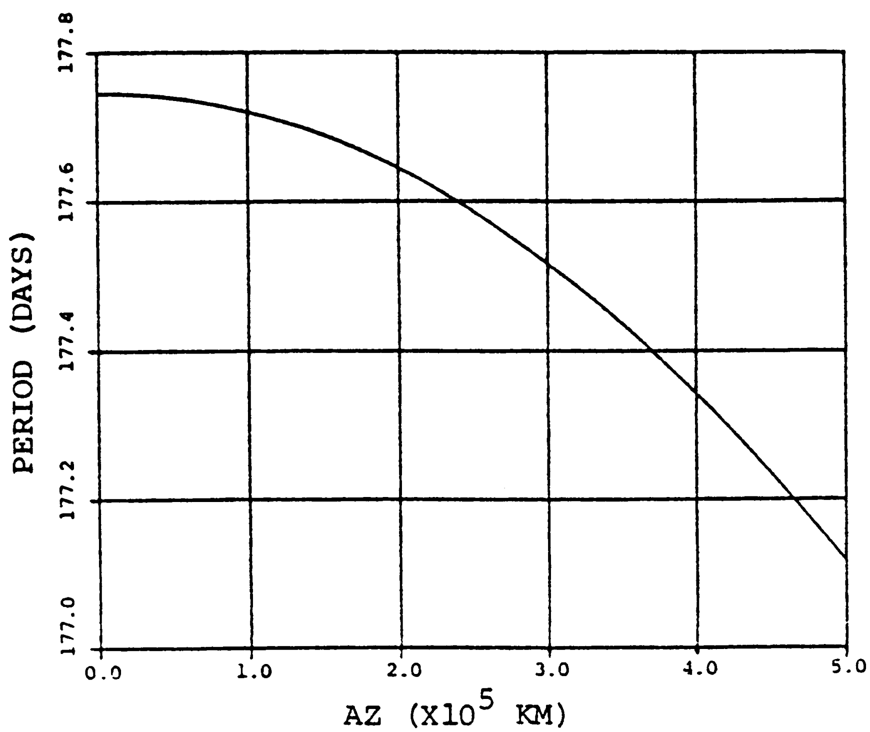
Fig. 3 Amplitude-Constraint Relationship

Halo Orbit Period Amplitude Relationship

- The halo orbit period T ($T = 2\pi/\lambda\omega$) can be computed as a function of A_z.
  - Amplitude constraint relationship: $l_1 A_x^2 + l_2 A_z^2 + \Delta = 0$.
  - Frequence connection ω ($\omega = 1 + \omega_1 + \omega_2 + \cdots$) with

$$\omega_1 = 0, \qquad \omega_2 = s_1 A_x^2 + s_2 A_z^2,$$

  - ISEE3 halo had a period of 177.73 days.



Differential Corrections

- While 3rd order approximations provide much insight, they are insufficient for serious study of motion near L_1.
- Analytic approximations must be combined with numerical techniques to generate an accurate halo orbit.
- This problem is well suited to a differential corrections process,
  - which incorporates the analytic approximations as the **first guess**
  - in an **iterative** process
  - aimed at producing initial conditions that lead to a halo orbit.

Differential Corrections: Variational Equations

- Recall 3D CR3BP equations:

$$\ddot{x} - 2\dot{y} = U_x \qquad \ddot{y} + 2\dot{x} = U_y \qquad \ddot{z} = U_z$$

  where $U = (x^2 + y^2)/2 + (1-\mu)d_1^{-1} + \mu d_2^{-1}$.

- It can be rewritten as 6 1st order ODEs: $\dot{\bar{x}} = f(\bar{x})$, where $\bar{x} = (x\ y\ z\ \dot{x}\ \dot{y}\ \dot{z})^T$ is the state vector.

- Given a reference solution $\bar{x}$ to ODE,
  - variational equations which are linearized equations for variations $\delta\bar{x}$ (relative to reference solution) can be written as

$$\dot{\delta\bar{x}}(t) = Df(\bar{x})\delta\bar{x} = A(t)\delta\bar{x}(t),$$

  where $A(t)$ is a matrix of the form

$$\begin{bmatrix} 0 & I_3 \\ \mathcal{U} & 2\Omega \end{bmatrix}.$$

Differential Corrections: Variational Equations

- Given a reference solution $\bar{x}$ to ODE,
  - variational equations can be written as

$$\dot{\delta\bar{x}}(t) = Df(\bar{x})\delta\bar{x} = A(t)\delta\bar{x}(t), \quad \text{where}$$

$$A(t) = \begin{bmatrix} 0 & I_3 \\ \mathcal{U} & 2\Omega \end{bmatrix}.$$

  - Matrix Ω can be written

$$\Omega = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  - Matrix $\mathcal{U}$ has the form

$$\mathcal{U} = \begin{bmatrix} U_{xx} & U_{xy} & U_{xz} \\ U_{yx} & U_{yy} & U_{yz} \\ U_{zx} & U_{zy} & U_{zz} \end{bmatrix},$$

    and is evalutated on reference solution.

Differential Corrections: State Transition Matrix

- Solution of variational equations is known to be of the form

$$\delta\bar{x}(t) = \Phi(t, t_0)\delta\bar{x}(t_0),$$

where $\Phi(t, t_0)$ represents state transition matrix from time t_0 to t.

  - State transition matrix reflects sensitivity of state at time t to small perturbations in initial state at time t_0.

- To apply **differential corrections**,
need to compute state transition matrix along a reference orbit.

- Since

$$\dot{\Phi}(t, t_0)\delta\bar{x}(t_0) = \dot{\delta}\bar{x}(t) = A(t)\delta\bar{x}(t) = A(t)\Phi(t, t_0)\delta\bar{x}(t_0),$$

we obtain ODEs for $\Phi(t, t_0)$:

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0),$$

with

$$\Phi(t_0, t_0) = I_6.$$

Differential Corrections: State Transition Matrix

- Therefore, state transition matrix along a reference orbit

$$\delta\bar{x}(t) = \Phi(t, t_0)\delta\bar{x}(t_0),$$

can be computed numerically
by integrating simultaneously the following 42 ODEs:

$$\begin{aligned} \dot{\bar{x}} &= f(\bar{x}), \\ \dot{\Phi}(t, t_0) &= A(t)\Phi(t, t_0), \end{aligned}$$

with initial conditions:

$$\begin{aligned} \bar{x}(t_0) &= \bar{x}_0, \\ \Phi(t_0, t_0) &= I_6. \end{aligned}$$

Numerical Computation of Halo Orbit

- Halo orbits are symmetric about xz-plane ($y = 0$).
  - They intersect this plan perpendicularly($\dot{x} = \dot{z} = 0$).
  - Thus, initial state vector take the form

$$\bar{x}_0 = (x_0 \; 0 \; z_0 \; 0 \; \dot{y}_0 \; 0)^T.$$

- Obtain 1st guess for $\bar{x}_0$ from 3rd order approximations.
  - ODEs are integrated until trajectory cross xz-plane.
  - For periodic solution, desired final state vector has the form

$$\bar{x}_f = (x_f \; 0 \; z_f \; 0 \; \dot{y}_f \; 0)^T.$$

  - While actual values for $\dot{x}_f, \dot{z}_f$ may not be zero,
    3 non-zero initial conditions ($x_0, z_0, \dot{y}_0$) can be used
    to drive these final velocities $\dot{x}_f, \dot{z}_f$ to zero.

Numerical Computation of Halo Orbit

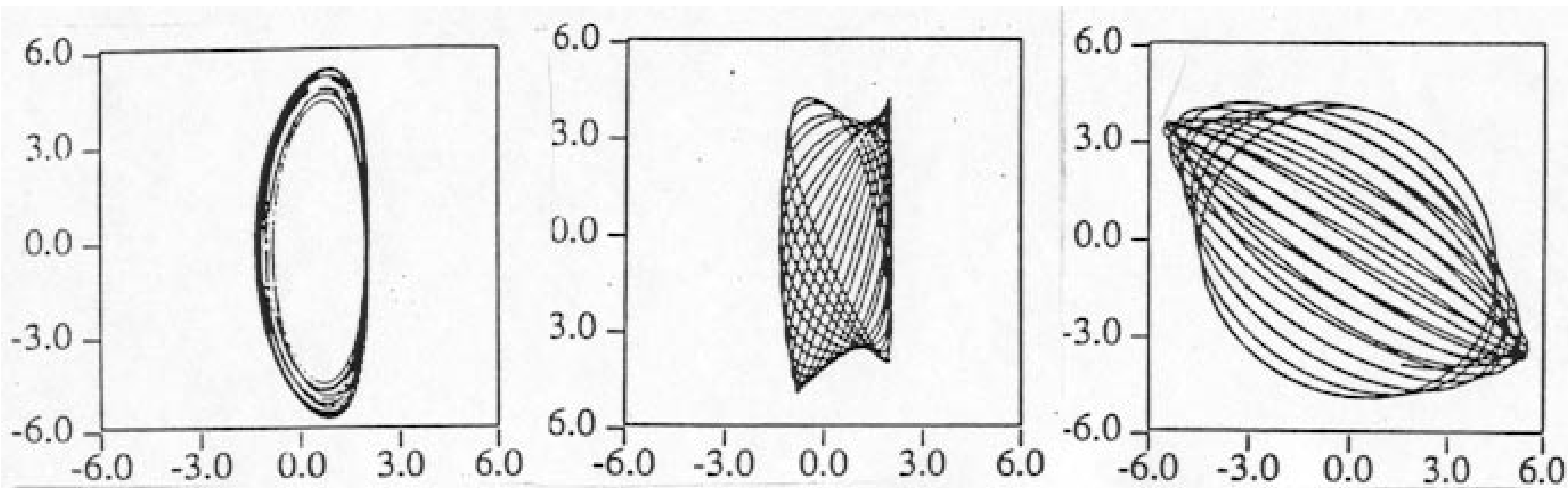
- Differential corrections use state transition matrix to change initial conditions

$$\delta \bar{x}_f = \Phi(t_f, t_0)\delta \bar{x}_0.$$

  - The change $\delta \bar{x}_0$ can be determined by the difference between actual and desired final states ($\delta \bar{x}_f = \bar{x}_f^d - \bar{x}_f$).
  - 3 initial states ($\delta x_0, \delta z_0, \delta \dot{y}_0$) are available to target 2 final states ($\delta \dot{x}_f, \delta z_f$).
  - But it is more convenient to set $\delta z_0 = 0$ and to use resulting 2×2 matrix to find $\delta x_0, \delta \dot{y}_0$.
- Similarly, the revised initial conditions $\bar{x}_0 + \delta \bar{x}_0$ are used to begin a second iteration.
- This process is continued until $\dot{x}_f = \dot{z}_f = 0$ (within some accptable tolerance).
  - Usually, convergence to a solution is achieved within 4 iterations.

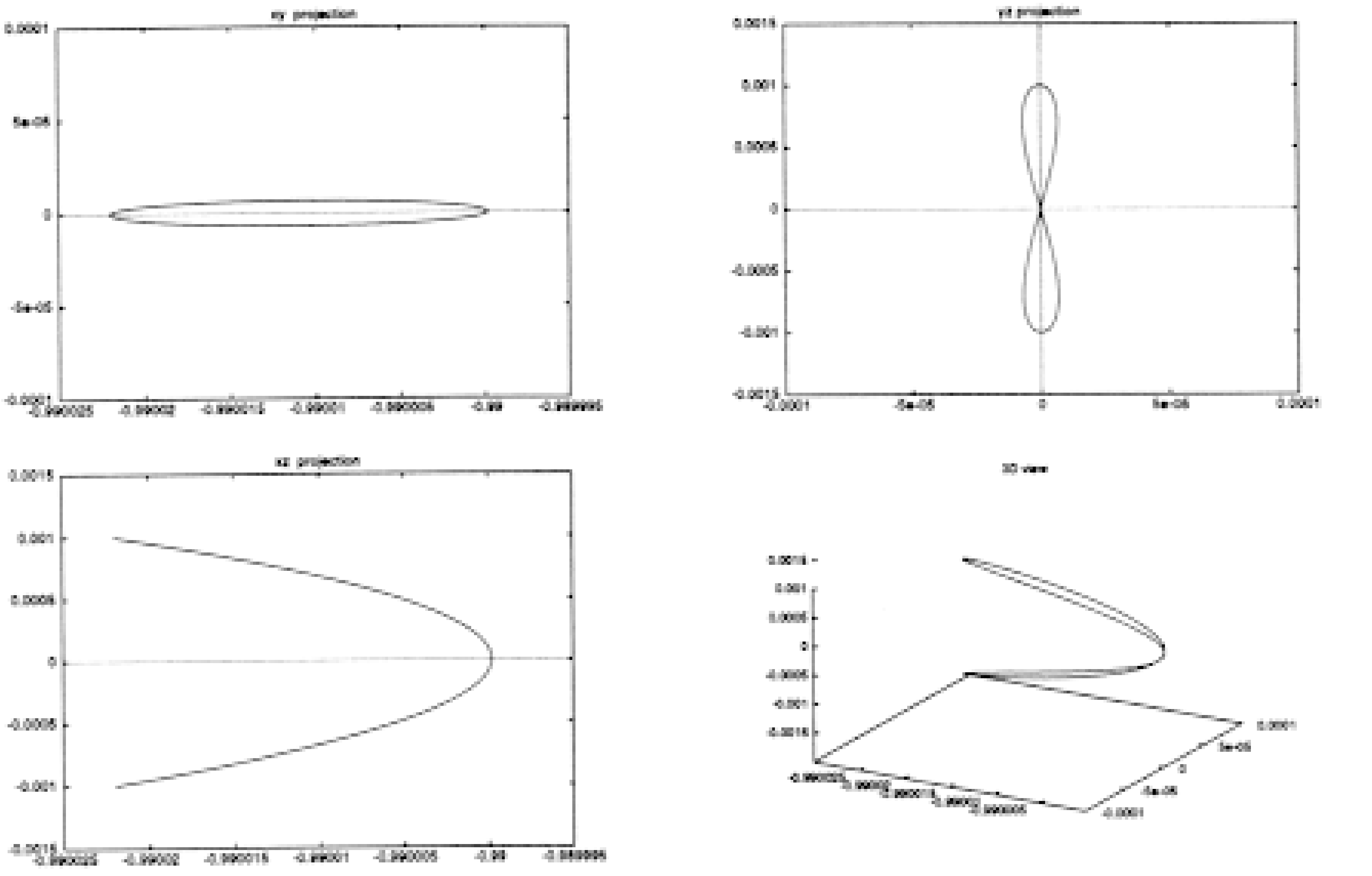
Numerical Computation of Lissajous Trajectories

- Howell and Pernicka [1987] used similar techniques (3rd order approximation and differential corrections) to compute lissajous trajectories.
- Gómez, Jorba, Masdemont and Simó [1991] used higher order expansions to compute halo, quasi-halo and lissajous orbits.



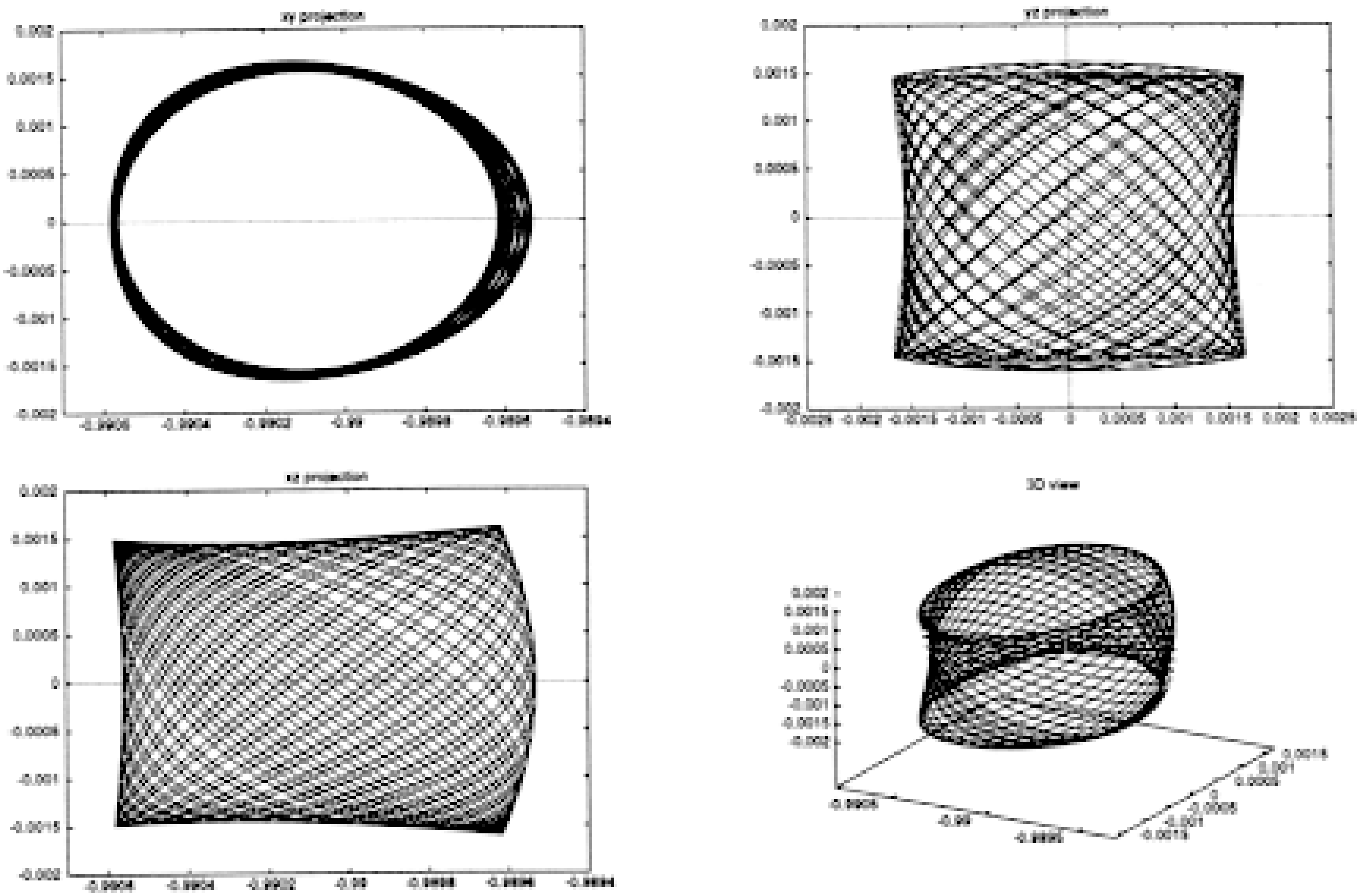
Veritcal Orbit

- A vertical orbit and its 3 projections.



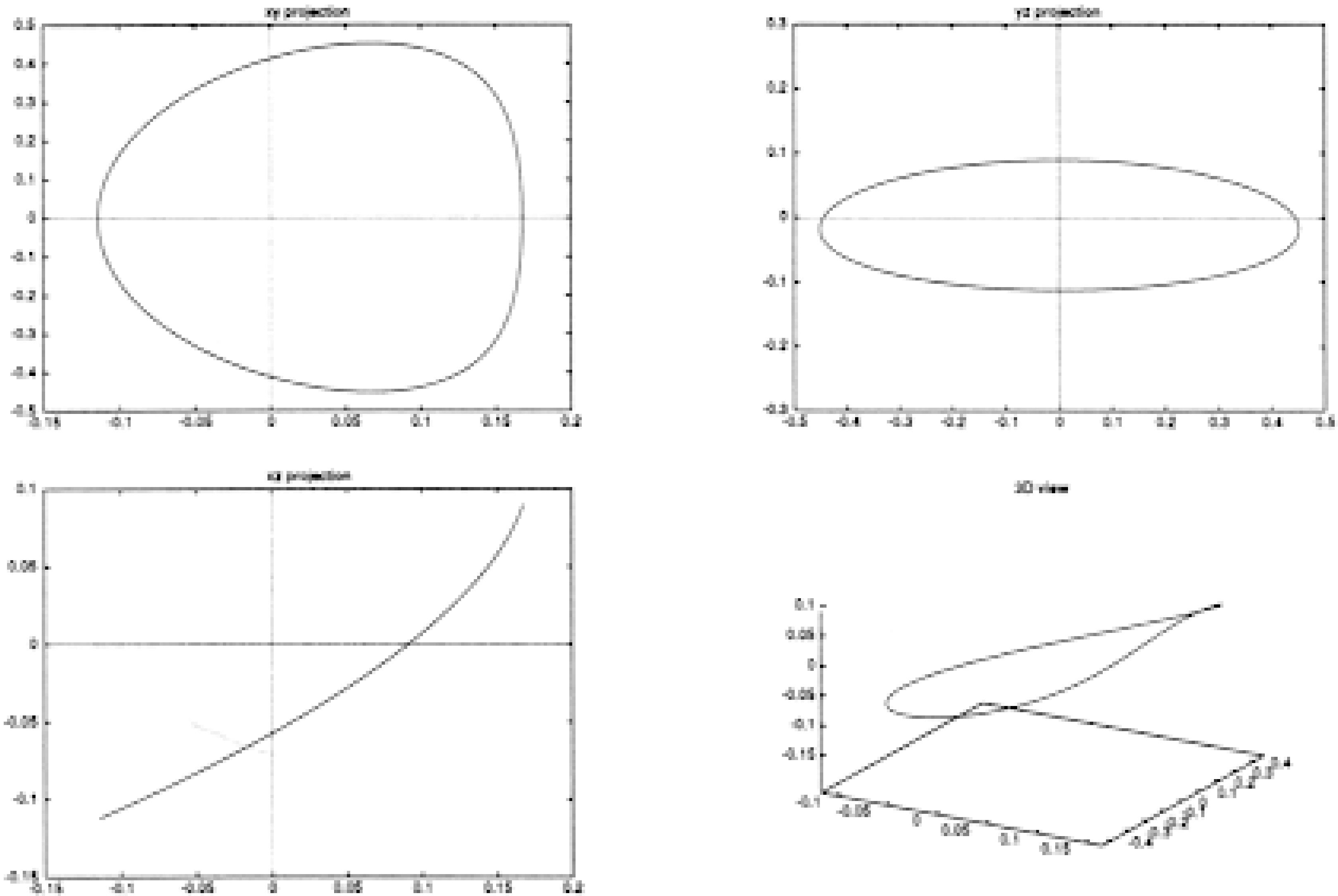
Lissajous Orbits

- A lissajous orbit and its 3 projections.



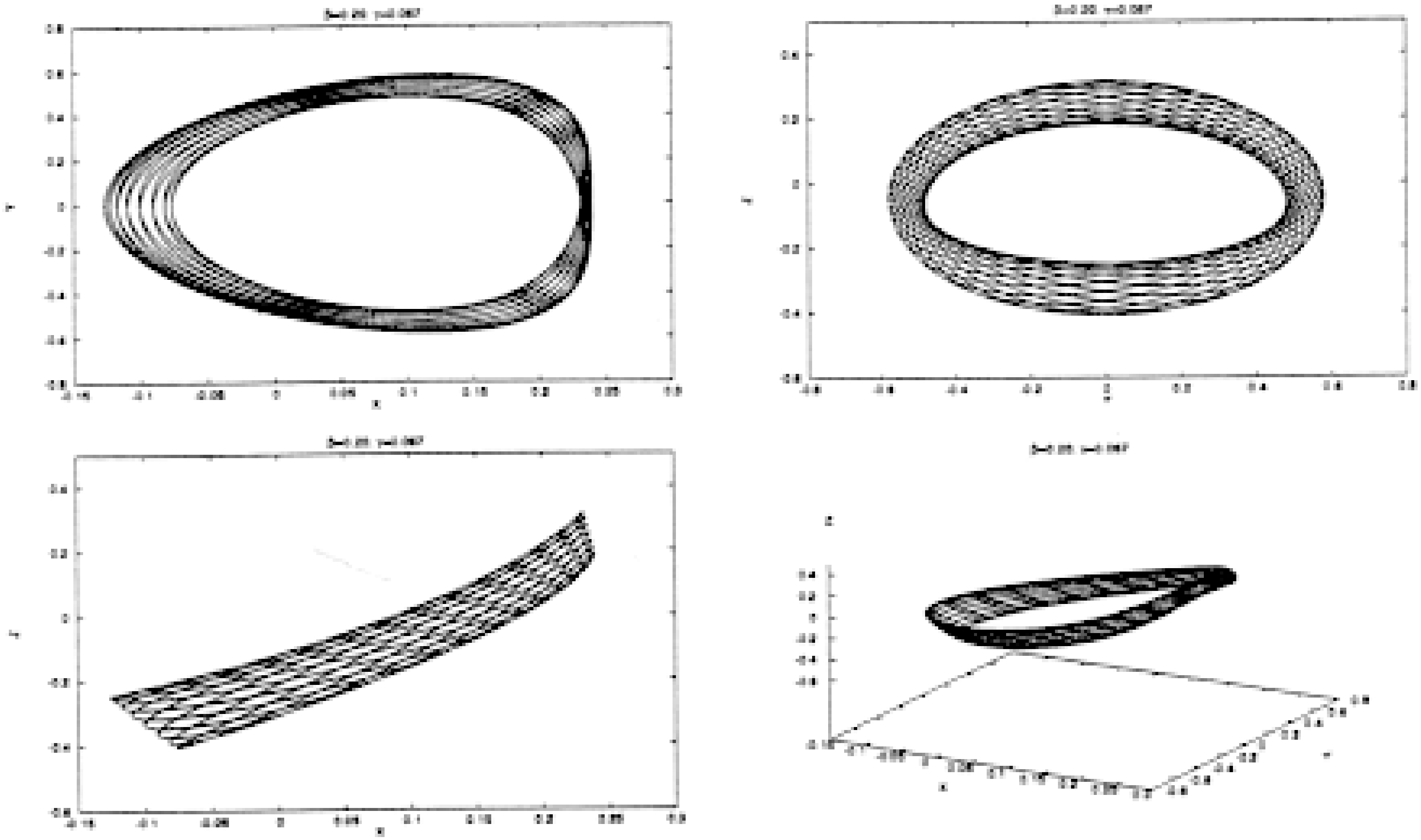
Halo Orbits

- A halo orbit and its 3 projections.



Quasi-Halo Orbits

- A quasi-halo orbit and its 3 projections.



Orbit Structure around L_1

- Poincaré sections of center manifold of L_1 corresponding to $h = 0.2, 0.5, 0.6, 1.0$.