# Dynamical Systems and Space Mission Design

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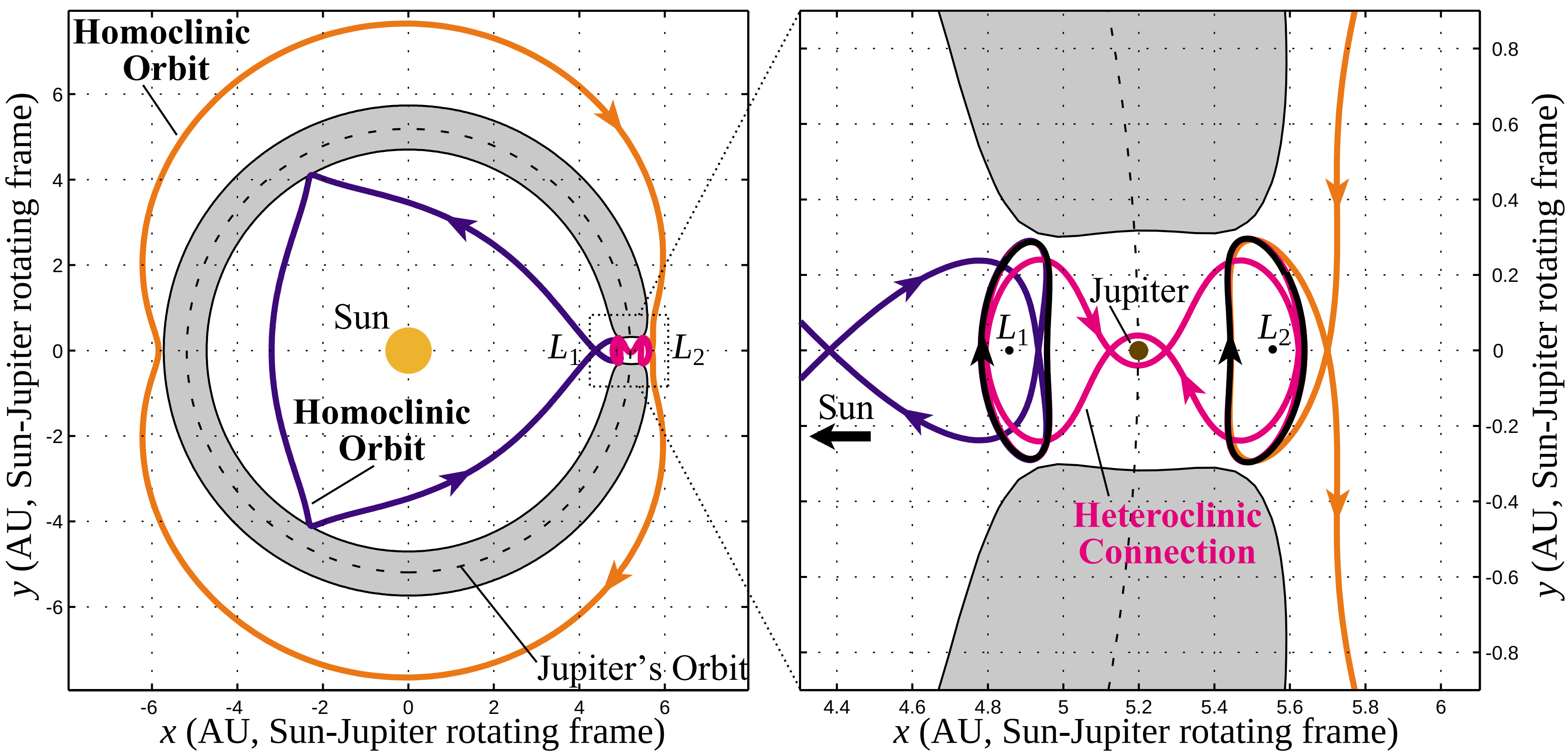
## Global Orbit Structure: Outline

- Outline of Lecture 3B:
  - Construction of Poincaré map.
  - Review of Horseshoe Dynamics.
  - Symbolic Dynamics for the PCR3BP.
  - Main Theorem on Global Orbit Structure.

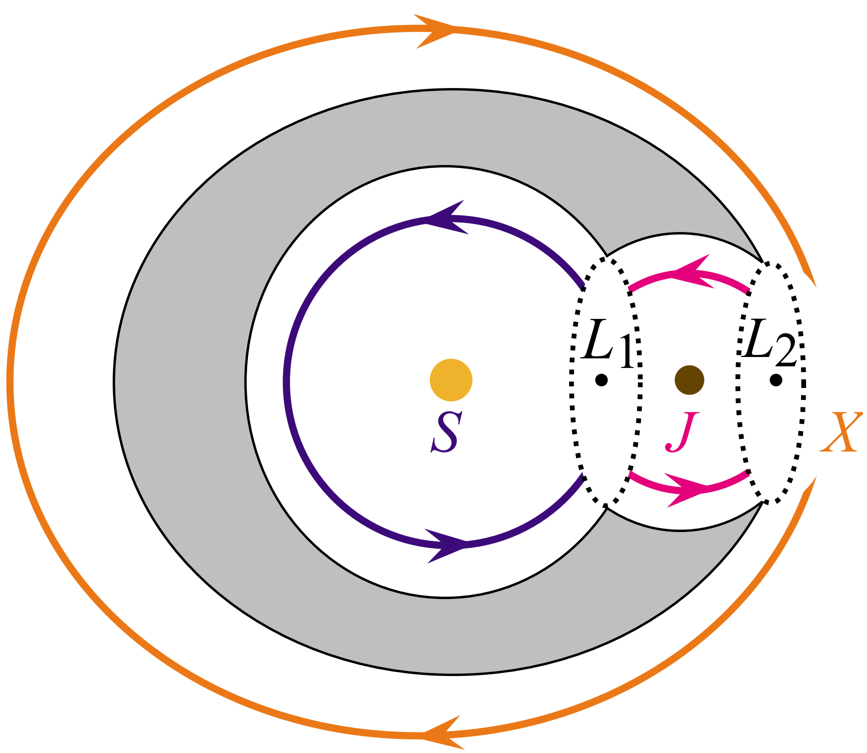
# Global Orbit Structure: Overview

- Found **heteroclinic connection** between pair of periodic orbits.
- Find a large class of **orbits** near this (homo/heteroclinic) ***chain***.
- Comet can follow these ***channels*** in rapid transition.



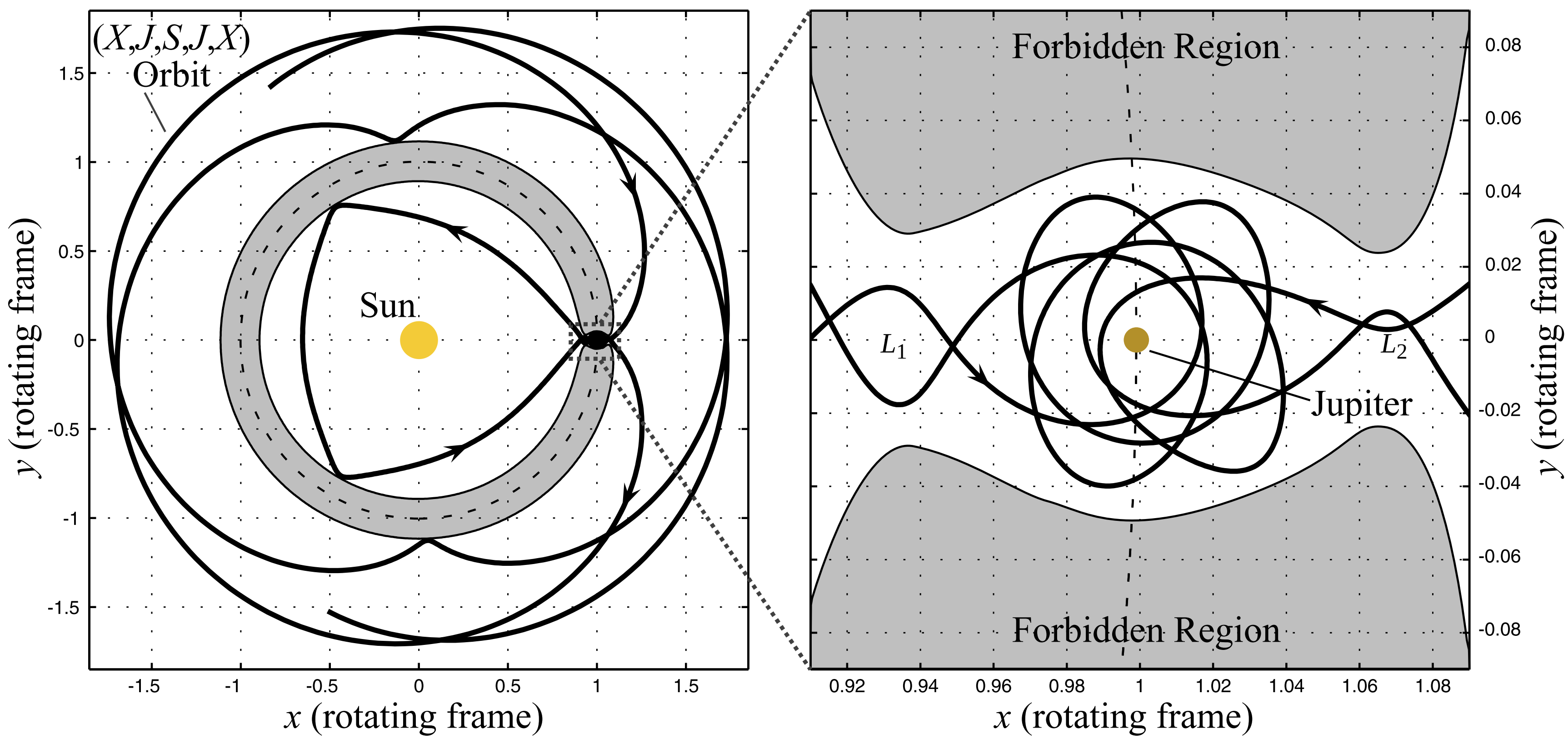
## Global Orbit Structure: Overview

- **Symbolic sequence** used to label itinerary of each comet orbit.
- ***Main Theorem***: For any admissible **itinerary**, e.g., (. . . , **X**, **J**, **S**, **J**, **X**, . . . ), there exists an orbit whose **whereabouts** matches this **itinerary**.
- Can even specify **number of revolutions** the comet makes around $L_1$ & $L_2$ as well as Sun & Jupiter.



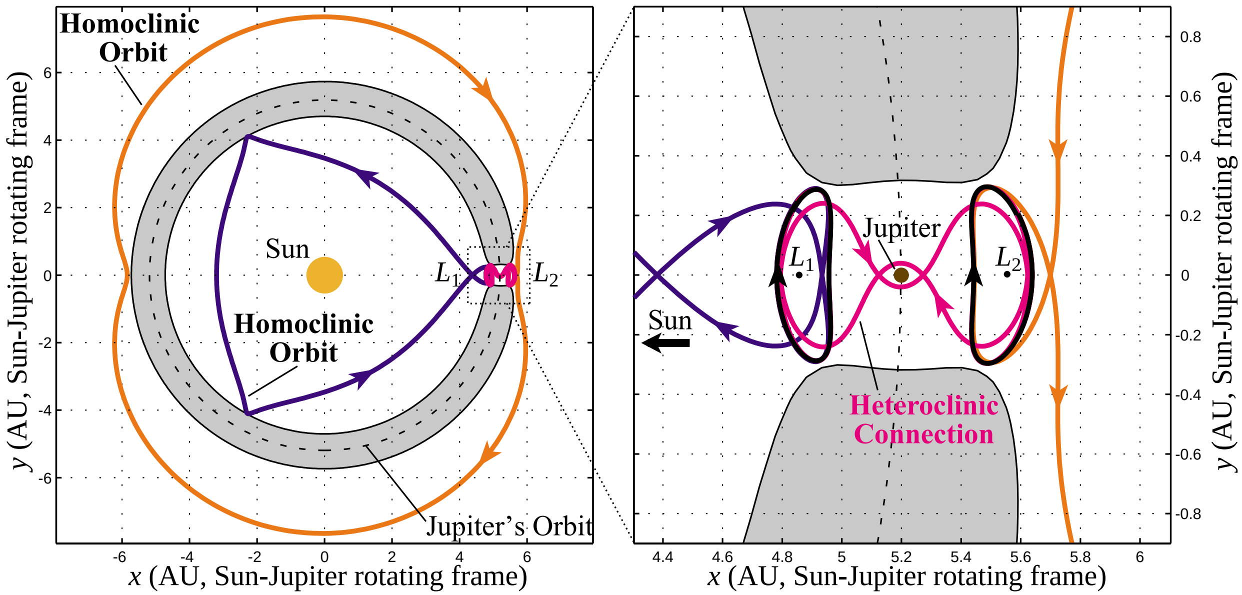
## Global Orbit Structure: Overview

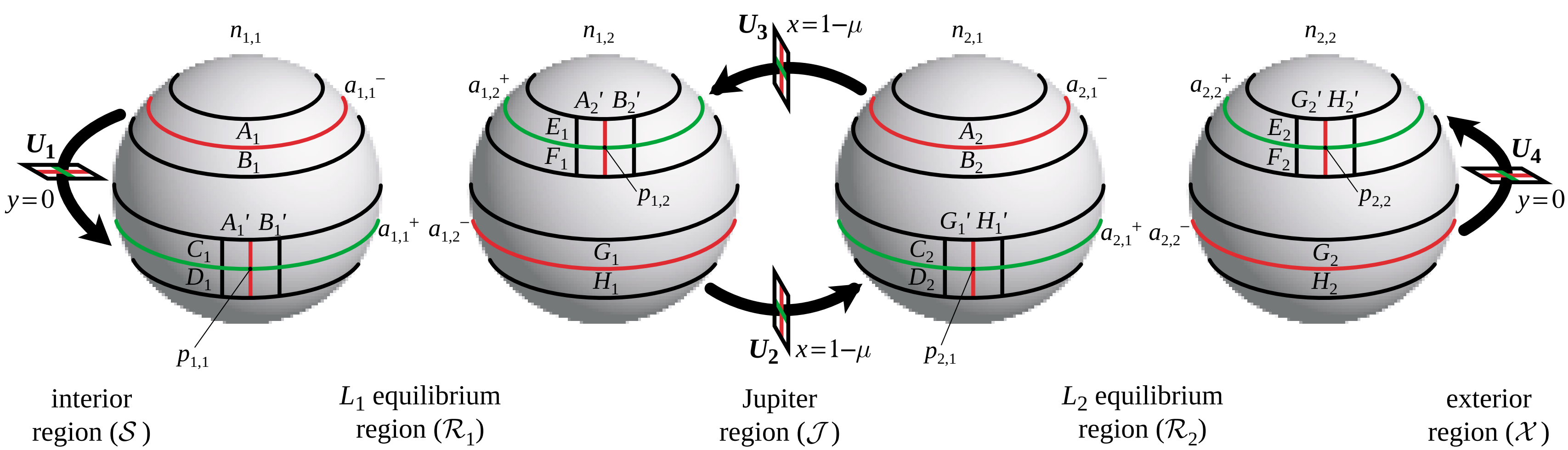
- Using the proof of **Main Theorem** as the guide, we develop procedure to construct orbit with **prescribed itinerary**.
- Example: An orbit with itinerary $(\mathbf{X}, \mathbf{J}; \mathbf{S}, \mathbf{J}, \mathbf{X})$.
- **Petit Grand Tour** of Jovian moons & **Shoot the Moon**.



## Global Orbit Structure: Energy Manifold

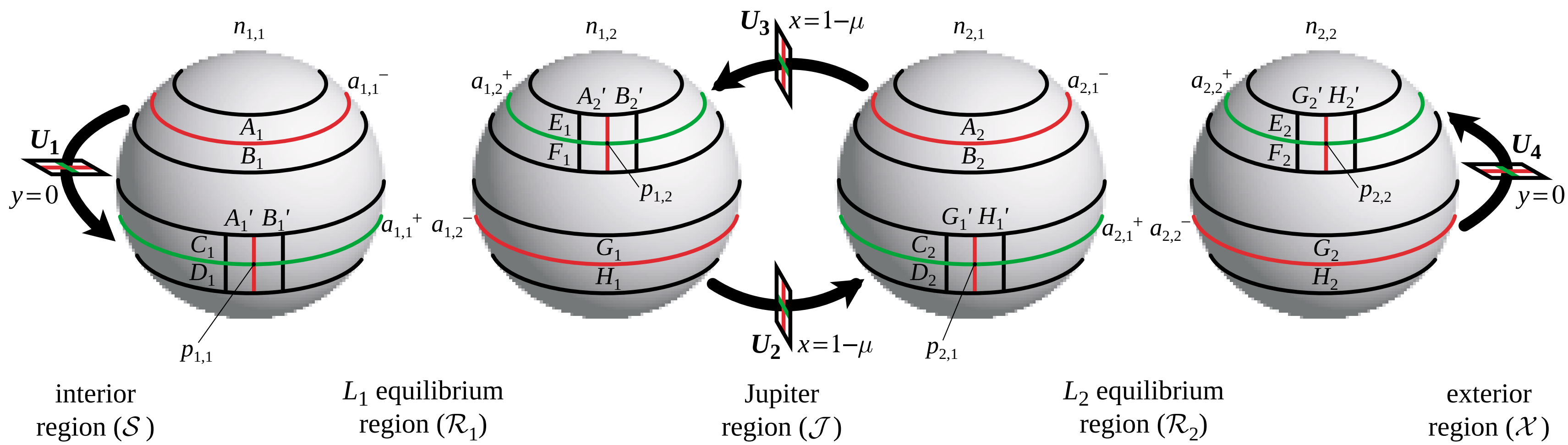
- Schematic view of energy manifold.

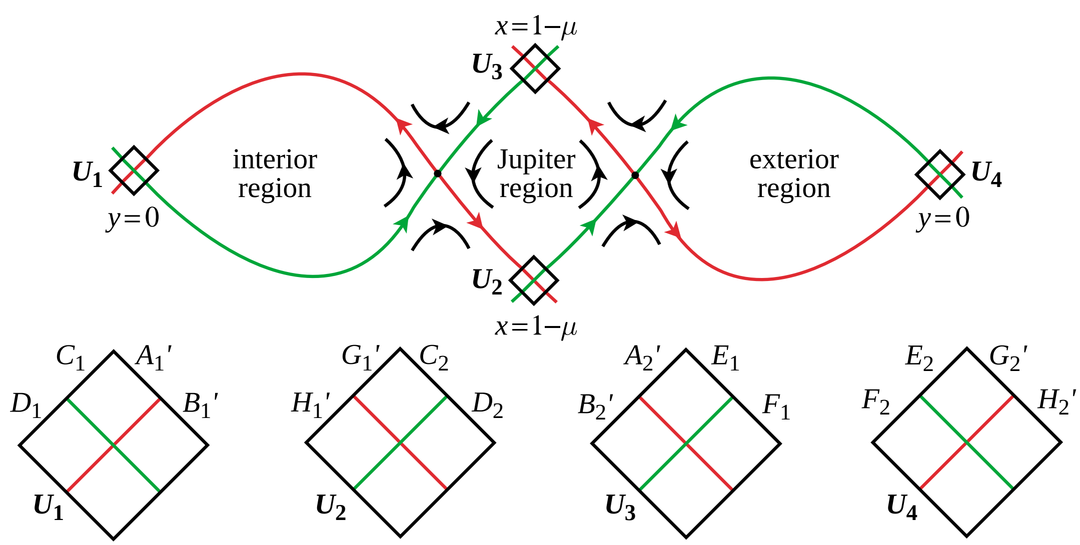


# Global Orbit Structure: Poincaré Map

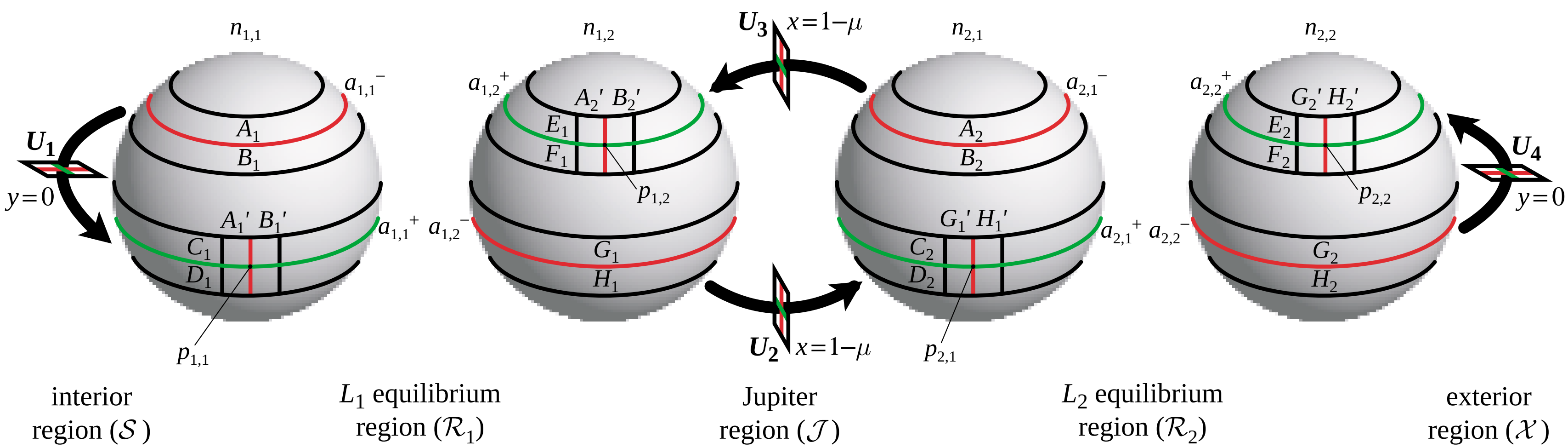
- Reducing study of global orbit structure to study of discrete map.



## Construction of Poincaré Map

- Construct **Poincaré map** $P$ (tranversal to the flow) whose domain $U$ consists of 4 squares $U_i$.
- Squares $U_1$ and $U_4$ contained in $y = 0$, each centers around a transversal **homoclinic** point.
- Squares $U_2$ and $U_3$ contained in $x = 1 - \mu$, each centers around a transversal **heteroclinic** point.



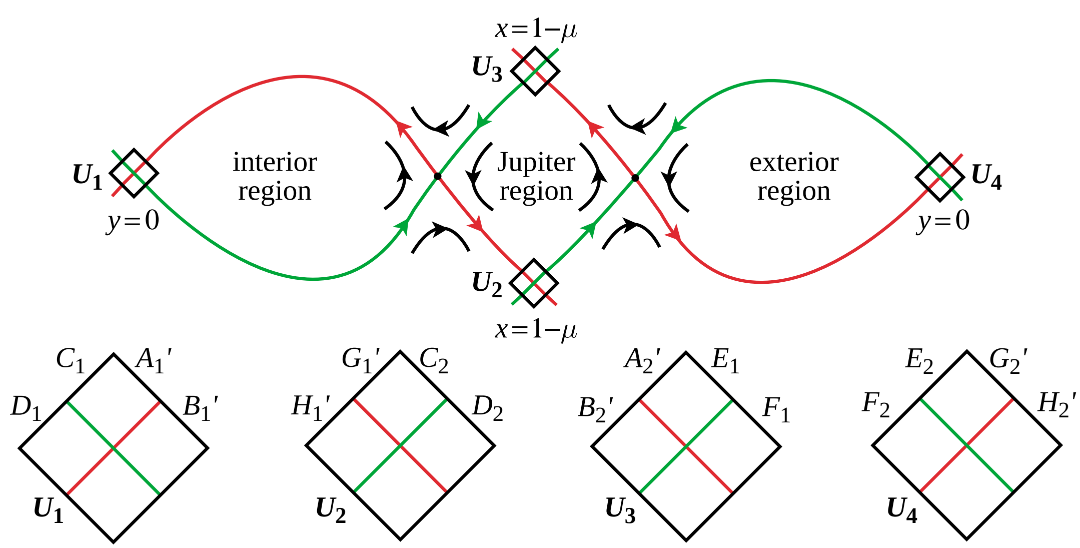
## Global Orbit Structure near the Chain

- Consider **invariant set** $\Lambda$ of points in $U$ whose images and pre-images under all iterations of $P$ remain in $U$.

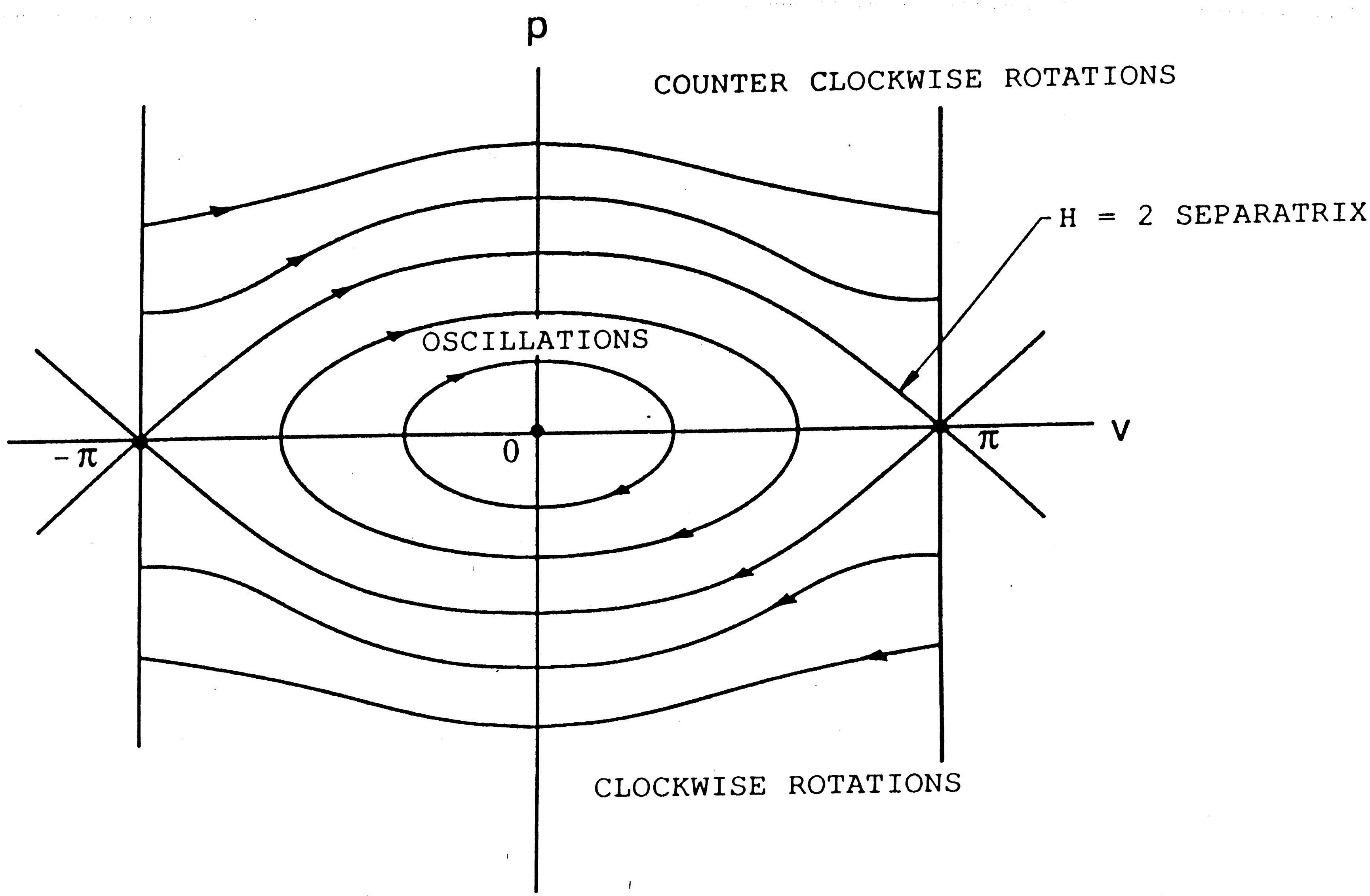
$$\Lambda = \cap_{n=-\infty}^{\infty} P^n(U).$$

- Invariant set $\Lambda$ contains all **recurrent orbits** near the chain.
  It provides insight into the **global dynamics** around the chain.
- Chaos theory told us to first consider only the **first** forward and backward iterations:

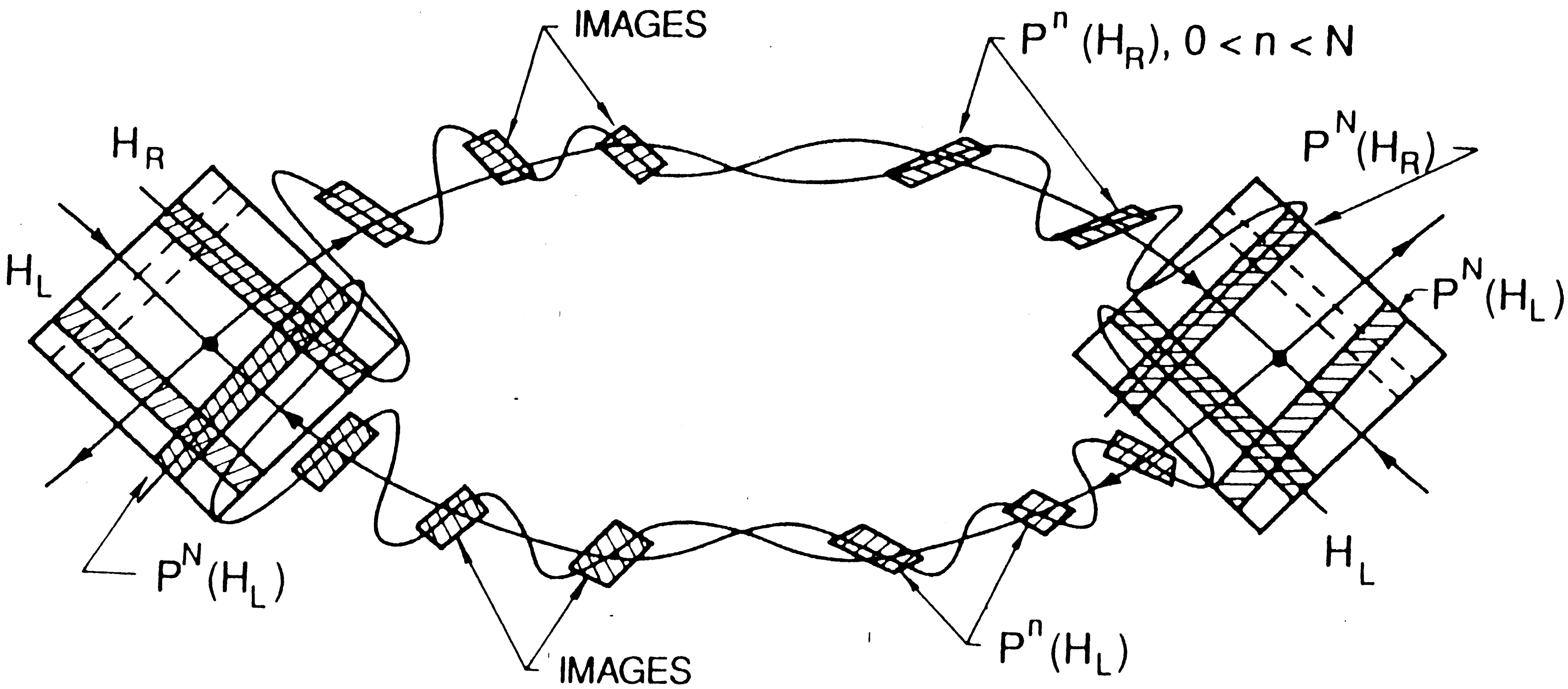
$$\Lambda^1 = P^{-1}(U) \cap U \cap P^1(U).$$



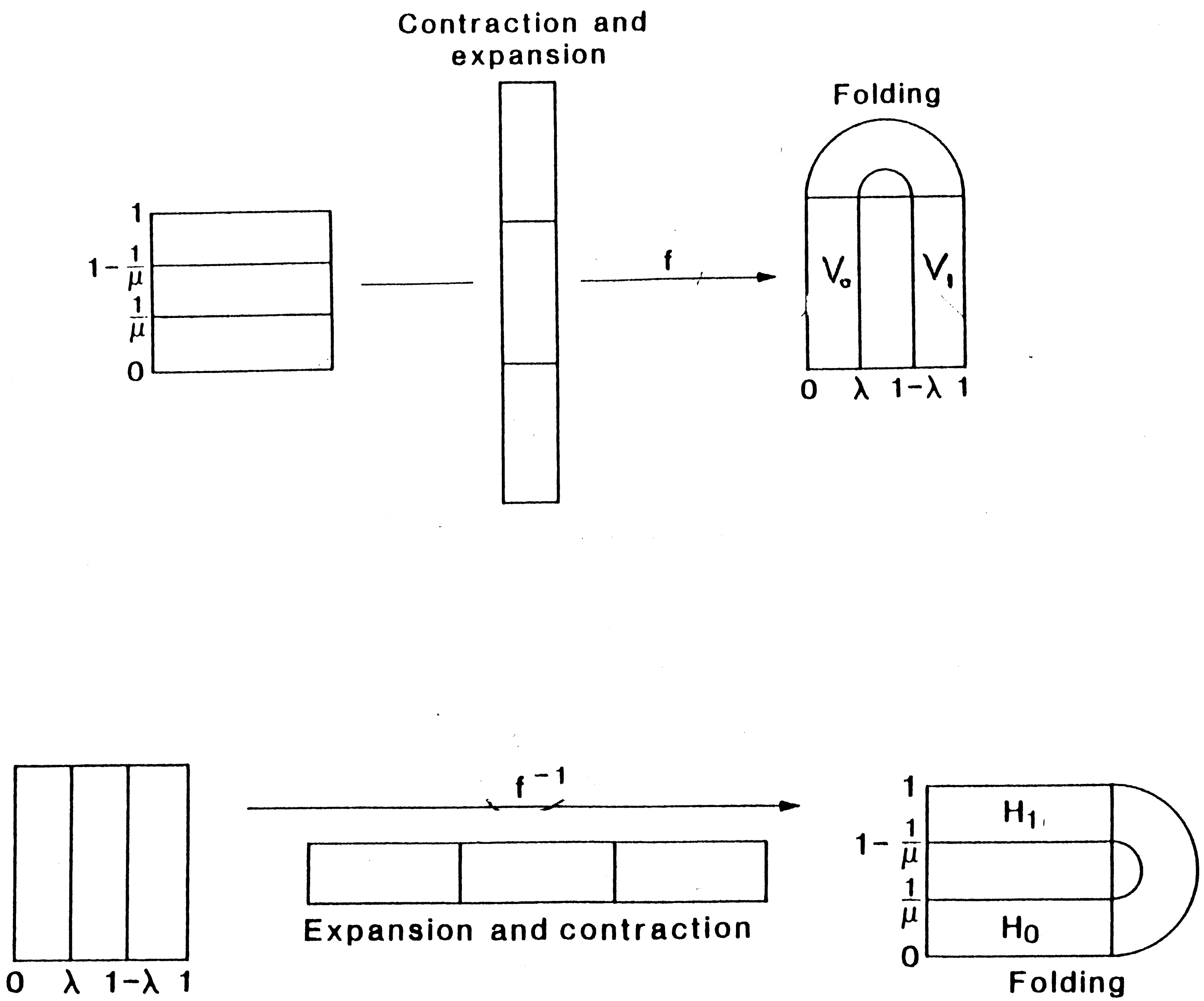
# Review of Horseshoe Dynamics: Pendulum



# Review of Horseshoe Dynamics: Forced Pendulum



# Review of Horseshoe Dynamics: First Iteration



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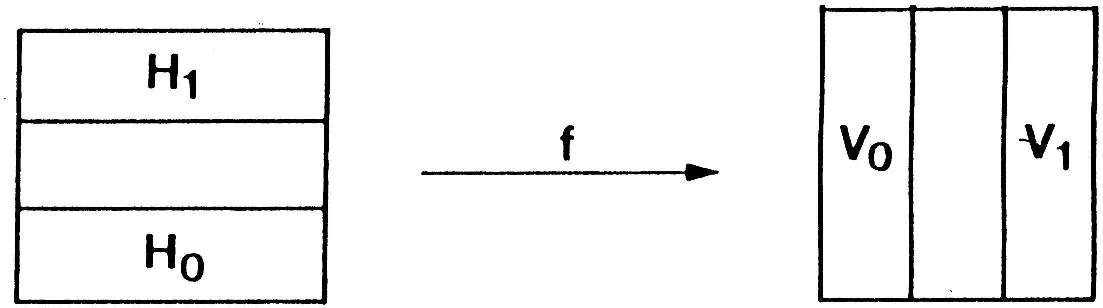
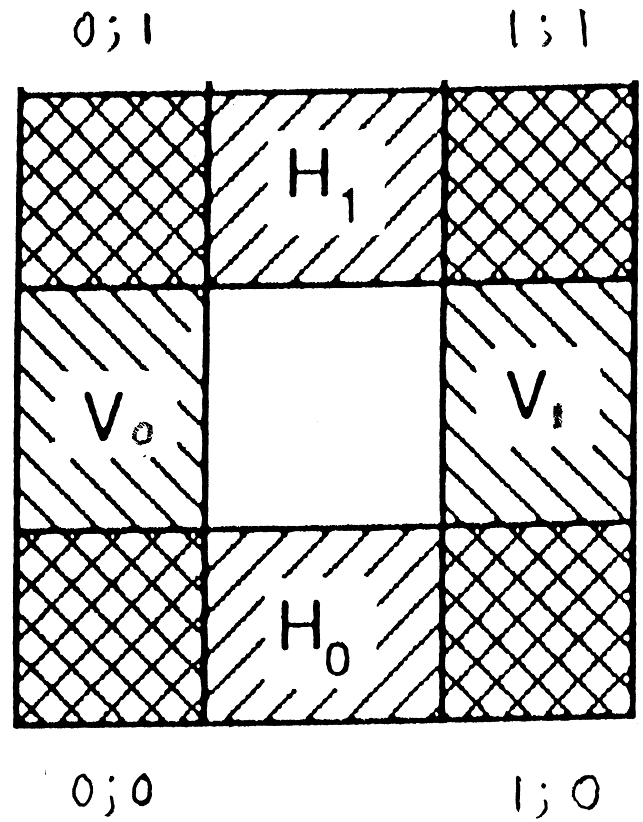
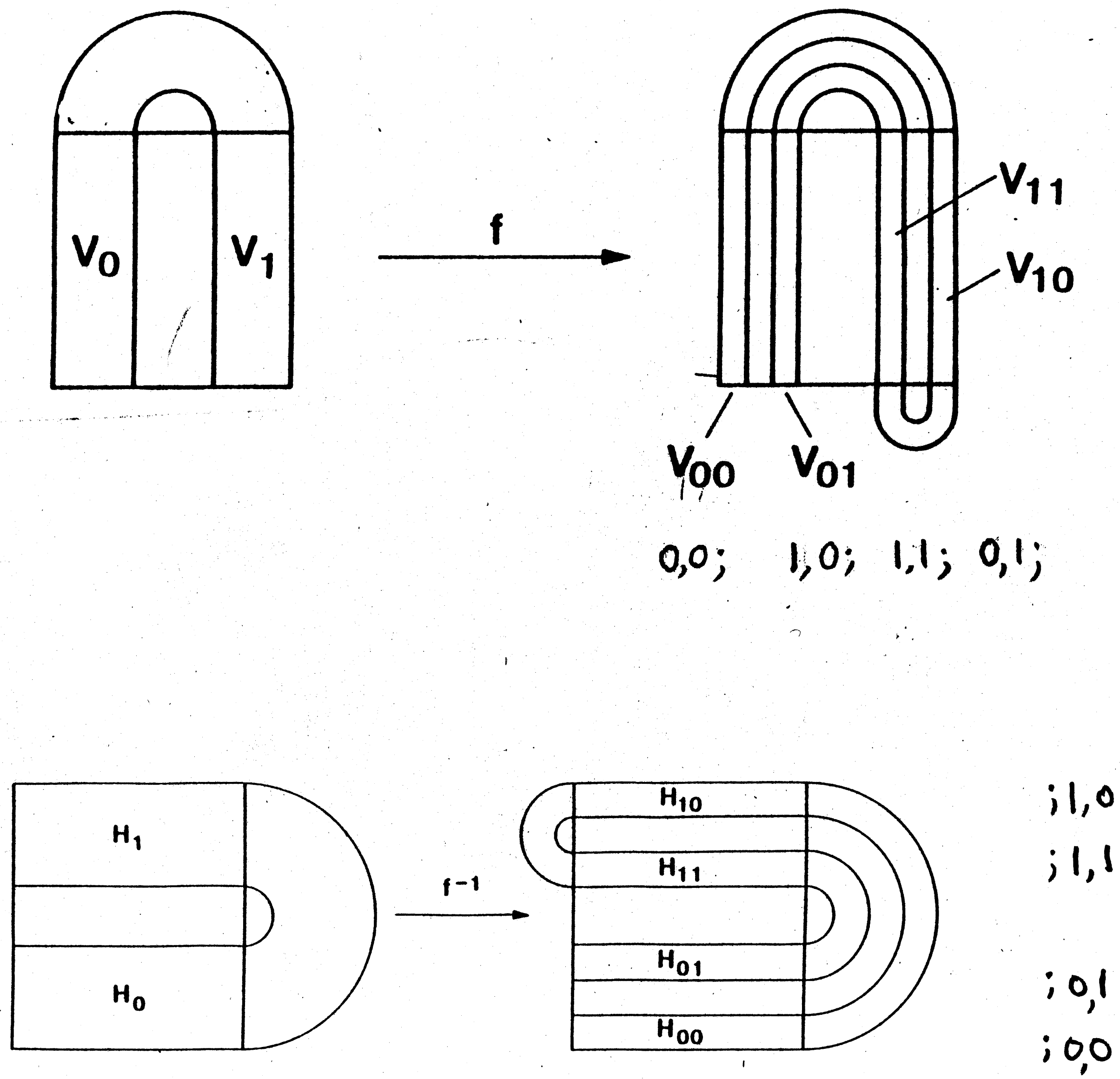
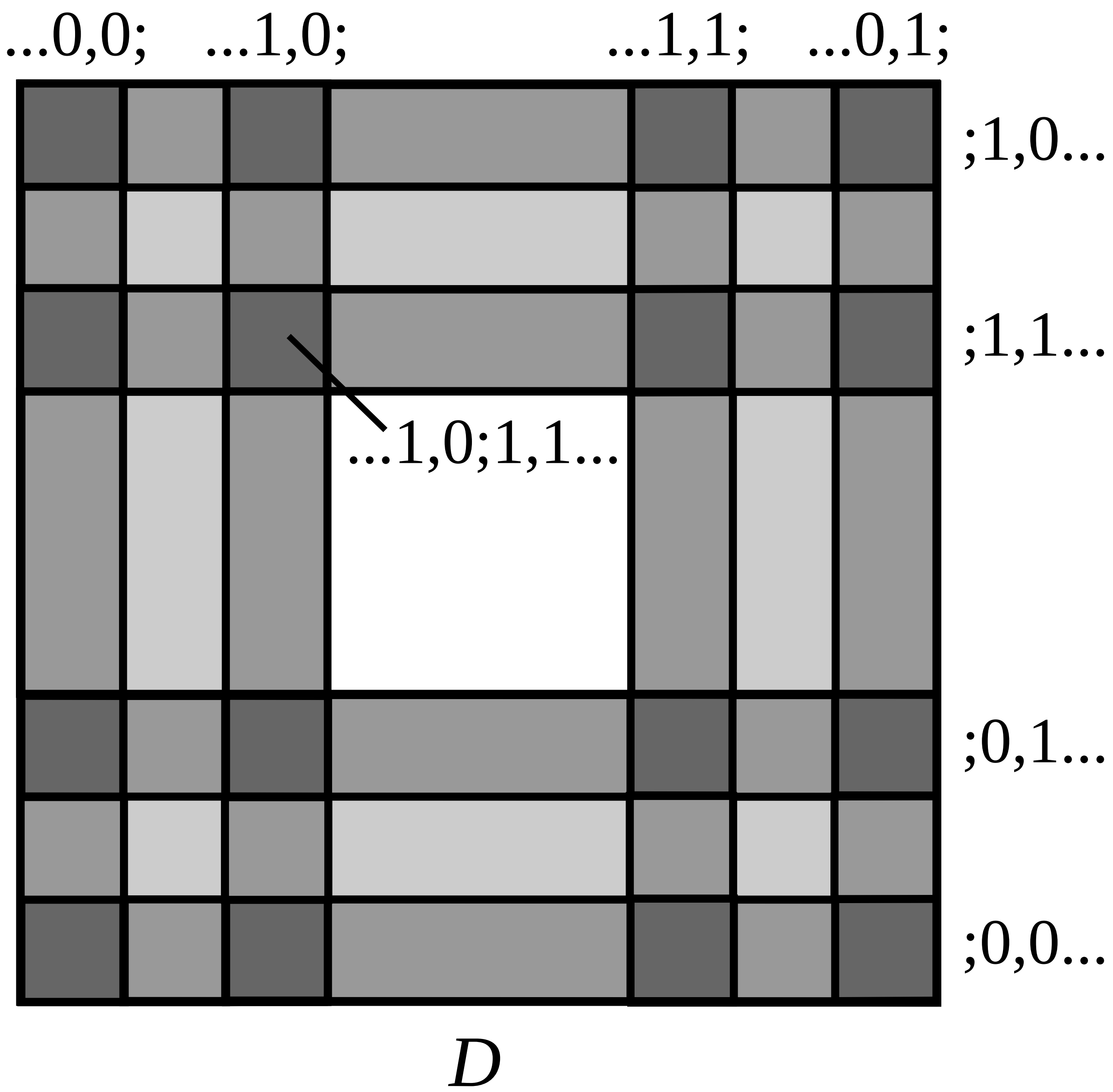


# Review of Horseshoe Dynamics: Second Iteration

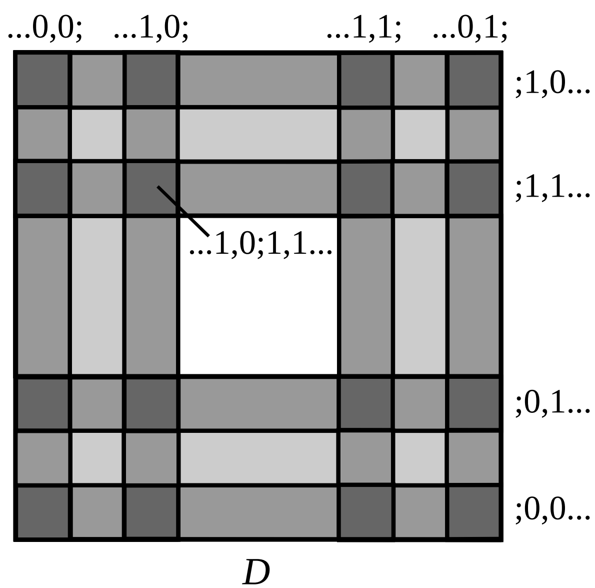


## Review of Horseshoe Dynamics: Second Iteration



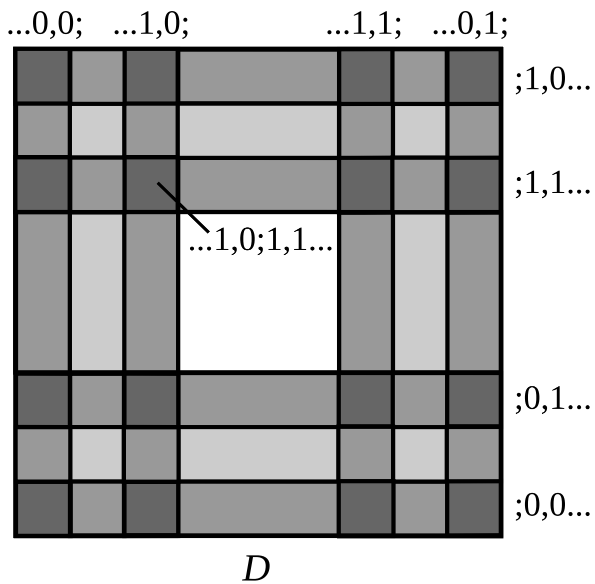
## Conley-Moser Conditions: Horseshoe-type Map

- For horseshoe-type map $h$ satisfying **Conley-Moser conditions**, the **invariant set** of all iterations, $\Lambda_h = \cap_{n=-\infty}^{\infty} h^n(Q)$, can be constructed and visualized in a standard way.
  - **Strip condition**: $h$ maps "horizontal strips" $H_0, H_1$ to "vertical strips" $V_0, V_1$, (with horizontal boundaries to horizontal boundaries and vertical boundaries to vertical boundaries).
  - **Hyperbolicity condition**: $h$ has uniform contraction in horizontal direction and expansion in vertical direction.



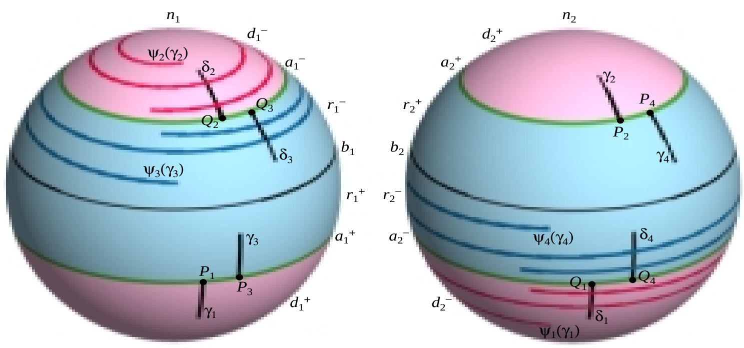
## Conley-Moser Conditions: Horseshoe-type Map

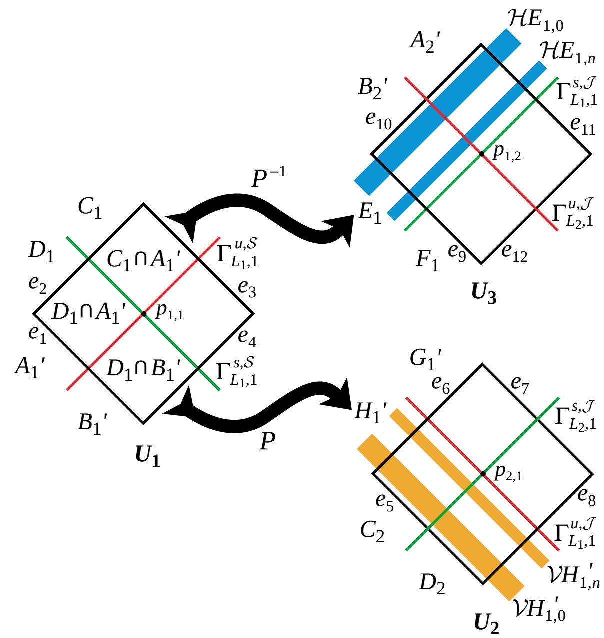
- **Invariant set** of first iterations $\Lambda_h^1 = h^{-1}(D) \cap D \cap h^1(D)$ has 4 squares, with addresses $(0;0), (1;0), (1;1), (0;1)$.
- **Invariant set** of second iterations has 16 squares **contained** in 4 squares of first stage.
- This process can be repeated **ad infinitum** due to **Conley-Moser Conditions**.
- What remains is **invariant set** of points $\Lambda_h$ which are in 1-to-1 corr. with set of bi-infinite sequences of 2 symbols $(\ldots, 0; 1, \ldots)$.



## Horizontal ($H_n^{31}$) & Vertical ($V_n^{21}$) Strips

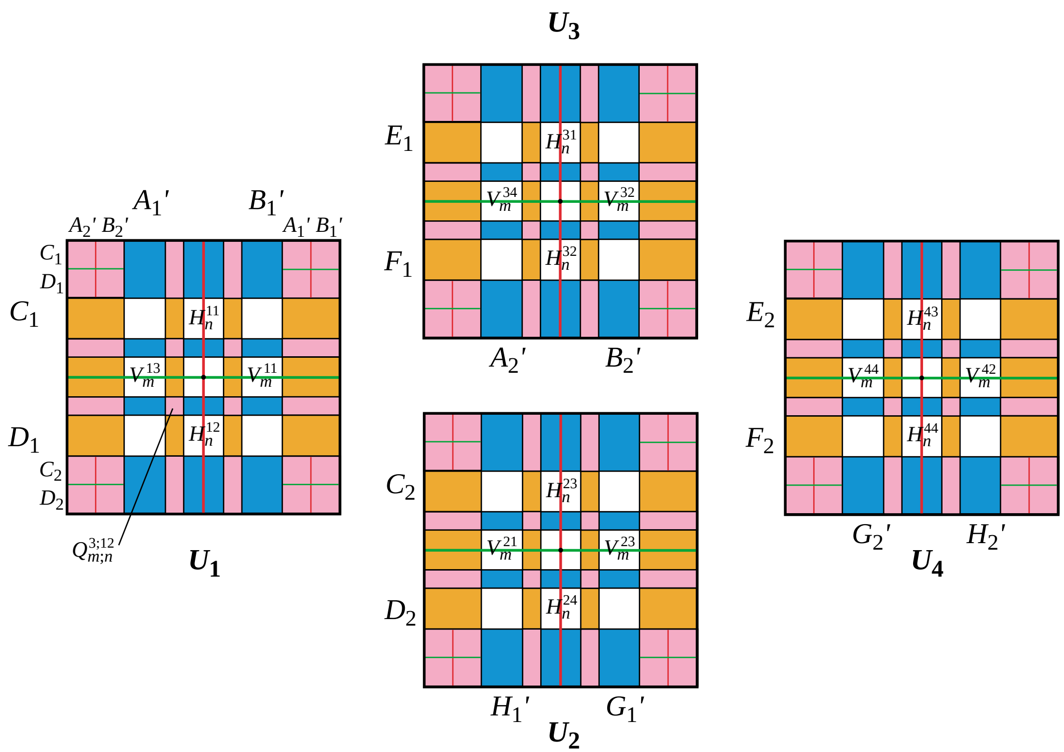
- Recall: image of **abutting arc** of **stable manifold cut** spiral infinitely many times towards **unstable manifold cut**.



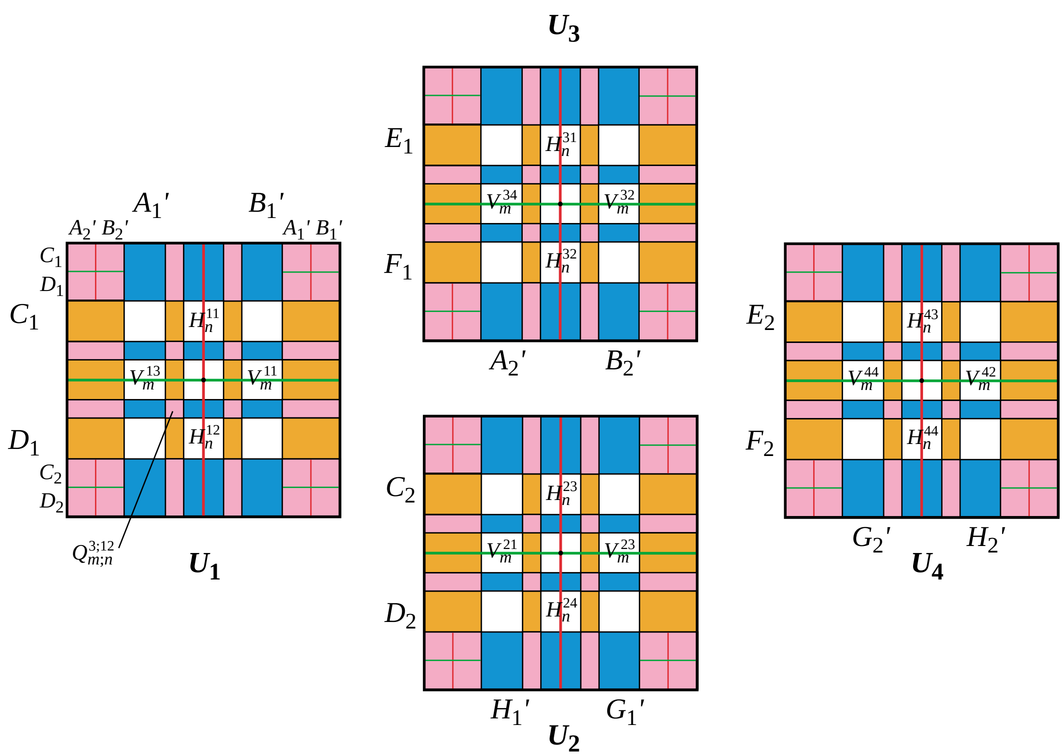
## Horizontal ($H_n^{ij}$) & Vertical ($V_m^{ji}$) Strips

- Hence, $U \cap P^{-1}(U)$ has 8 families of horizontal strips $H_n^{ij}$.
  - Each point in **horizontal strip** $H_n^{ij}$ is in $U_i$ and will wind $n$ times around an equilibrium point before reaching $U_j$.
  - We can associate each point in $H_n^{ij}$ both an **address** in $U$ and an **itinerary** $(;u_i, n, u_j)$.



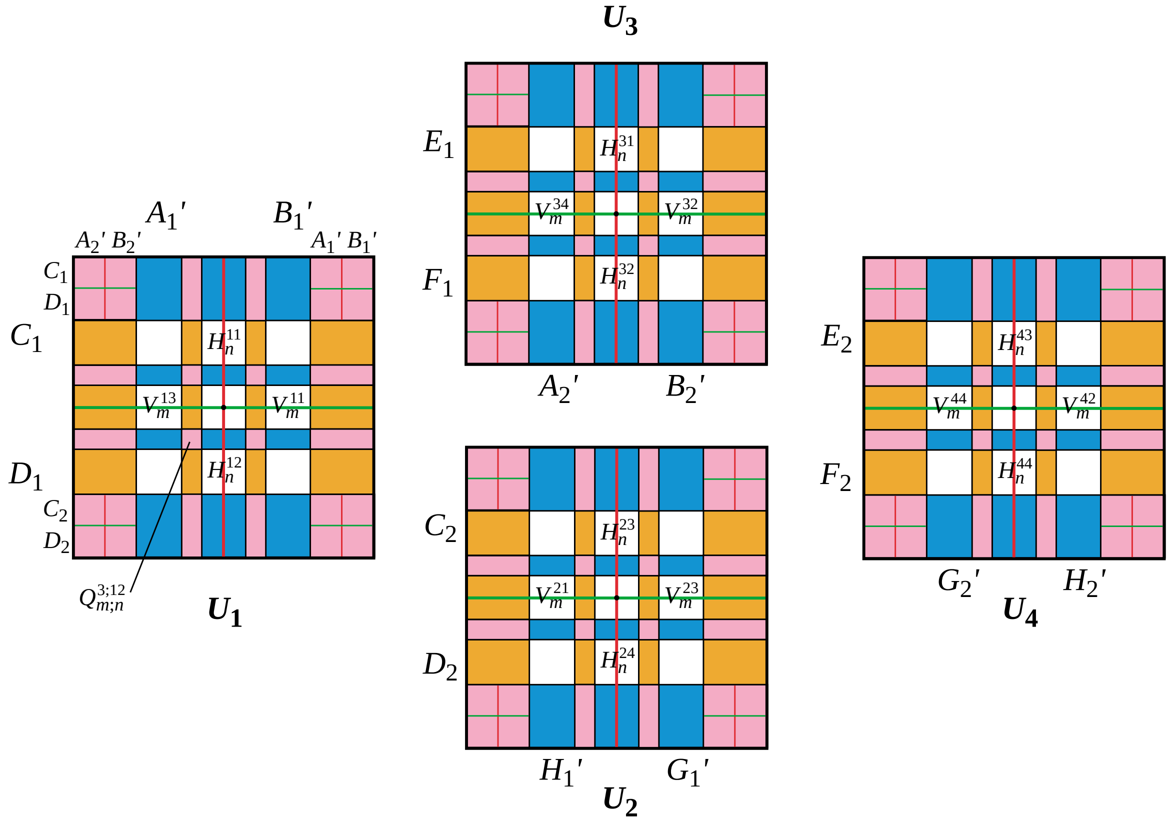
## Horizontal ($H_n^{ij}$) & Vertical ($V_m^{ji}$) Strips

- Similarly, $U \cap P^1(U)$ consists of 8 families of vertical strips $V_m^{ji}$.
  - Each point in **vertical strip** $V_m^{ji}$ came from $U_i$ and has wound $m$ times around an equilibrium point before arriving at $U_j$.
  - We can associate each point in $V_m^{ji}$ both an **address** in $U$ and an **itenerary** $(u_i, m; u_j)$.



## Invariant Set $\Lambda^1$ under First Interations

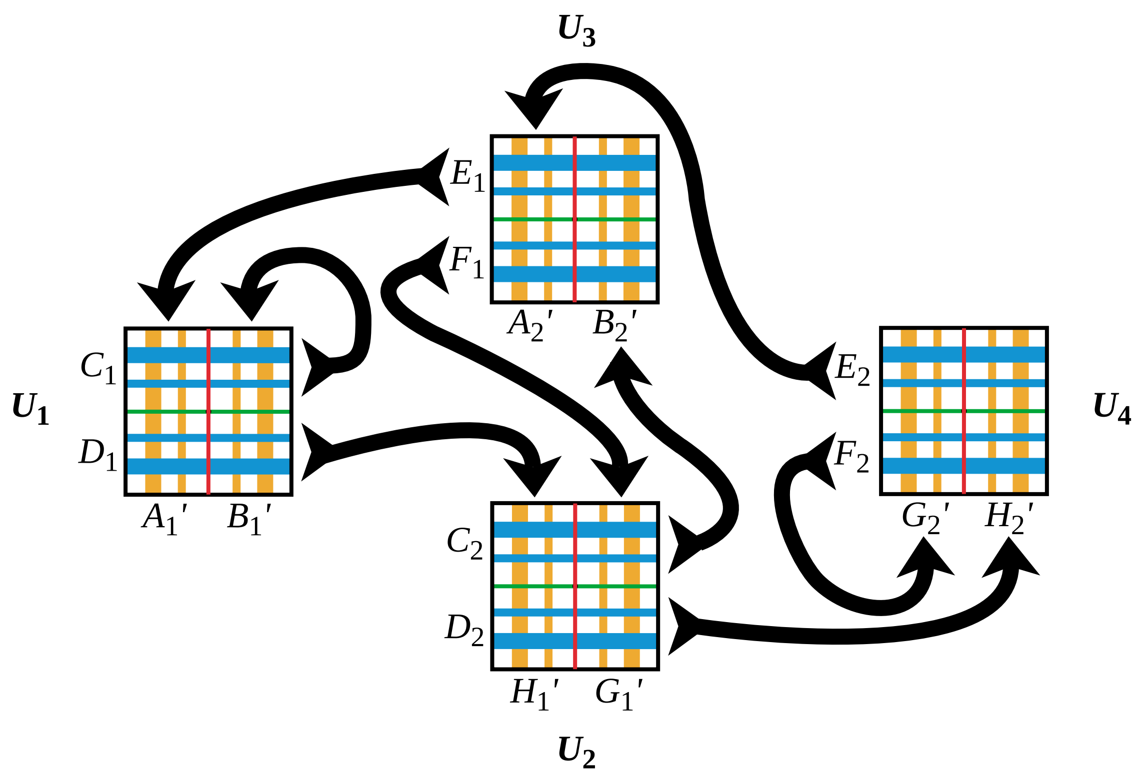
- The set $\Lambda^1 = P^{-1}(U) \cap U \cap P^1(U)$ is intersections of all **horizontal** and **vertical** strips.
  - Each **point** of $Q^{3;12}_{m;n} = H^{12}_n \cap V^{13}_m$ has an **itenerary** $(u_3, m; u_1, n, u_2)$ which is a concatenation of $(u_1; n, u_2)$ $(H^{12}_n)$ and $(u_3, m; u_1)$ $(V^{13}_m)$.



## Application of Symbolic Dynamics

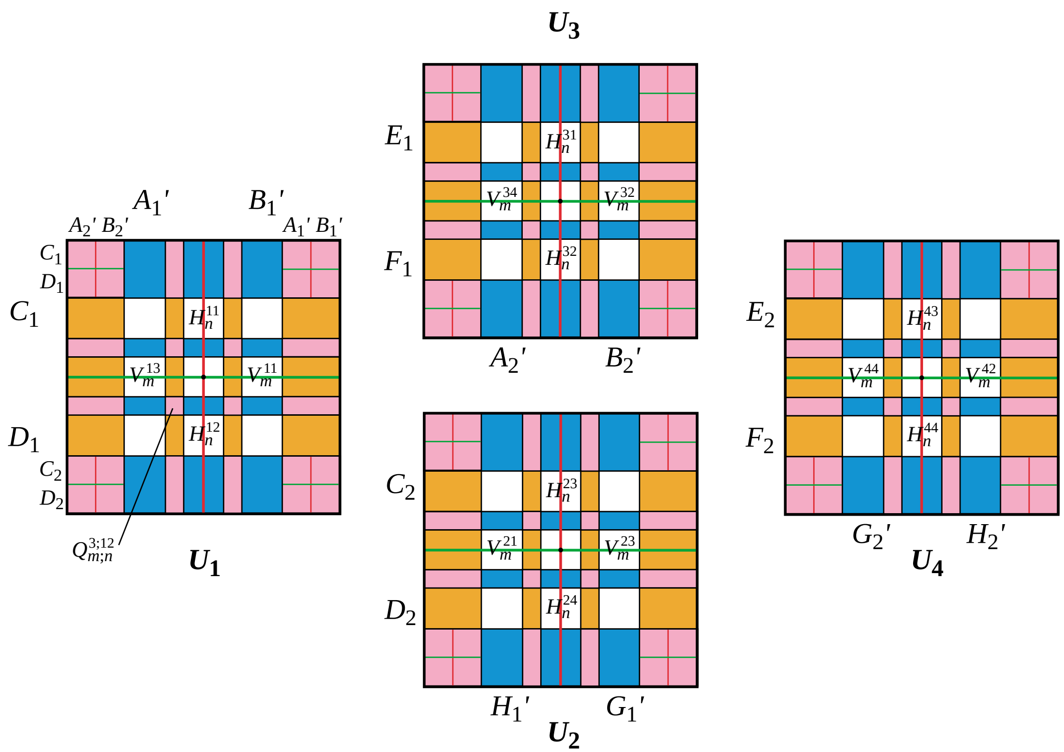
- Labeling "squares" $Q^{3;12}_{m;n}$ with $(u_3, m; u_1, n, u_2)$ is in line with characterizing orbits via bi-infinite sequences of "symbols".
- To keep track of **itenerary** w.r.t. 4 squares $U_i$, we use **subshift** with 4 symbols $u_i$, $(\ldots, u_3; u_1, u_2, \ldots)$, and a transition matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$



## Application of Symbolic Dynamics

- To keep track of **number of revolutions** around $L_1$ or $L_2$, we use full shift with integer symbols, $(\ldots, m; n, \ldots)$.
- "Squares" $Q^{i;jk}_{m;n}$ in 1-to-1 corr. with sequences $(u_i, m; u_j, n, u_k)$.
- **Symbolic sequence** is used to label **address** of each "square" and identifies **itenerary** of its orbits.
- To generalize beyond first iteration, need to review chaos theory.



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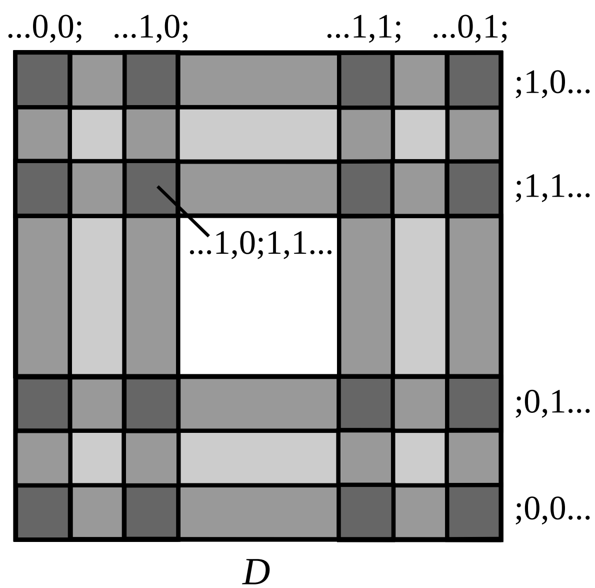

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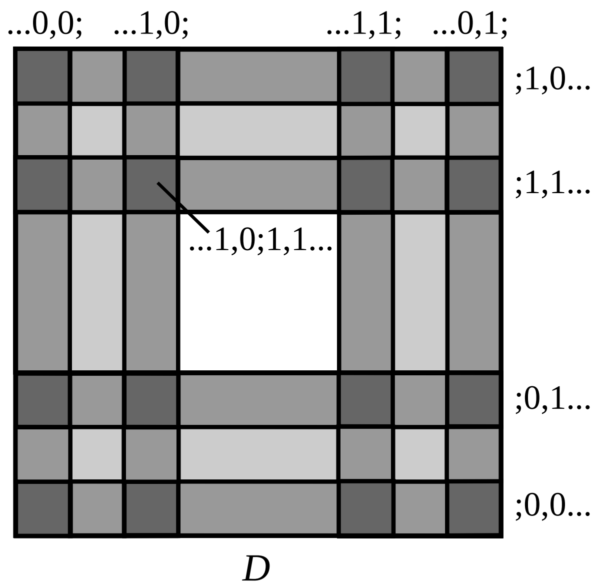
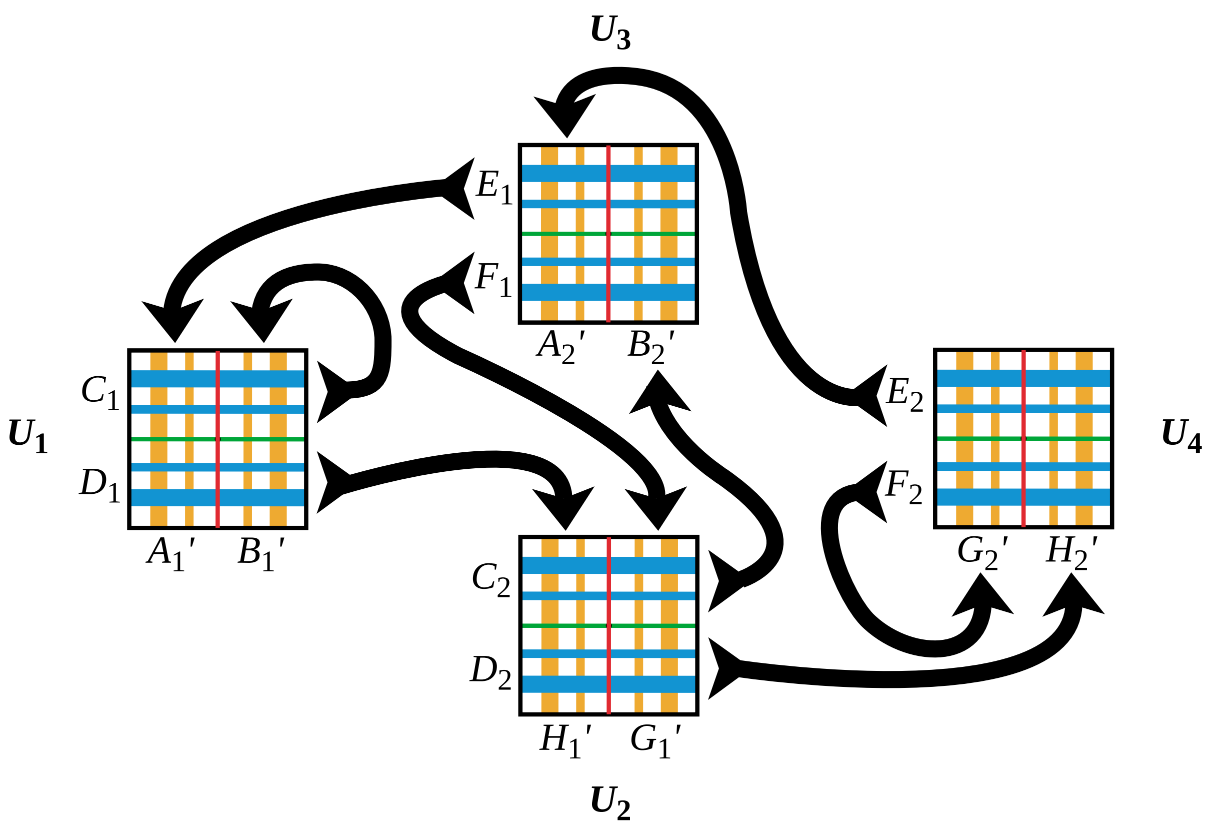
- This process can be repeated **ad infinitum** due to **Conley-Moser Conditions**.
- What remains is **invariant set** of points $\Lambda_h$ which are in 1-to-1 corr. with set of bi-infinite sequences of 2 symbols $(\ldots, 0; 1, \ldots)$.



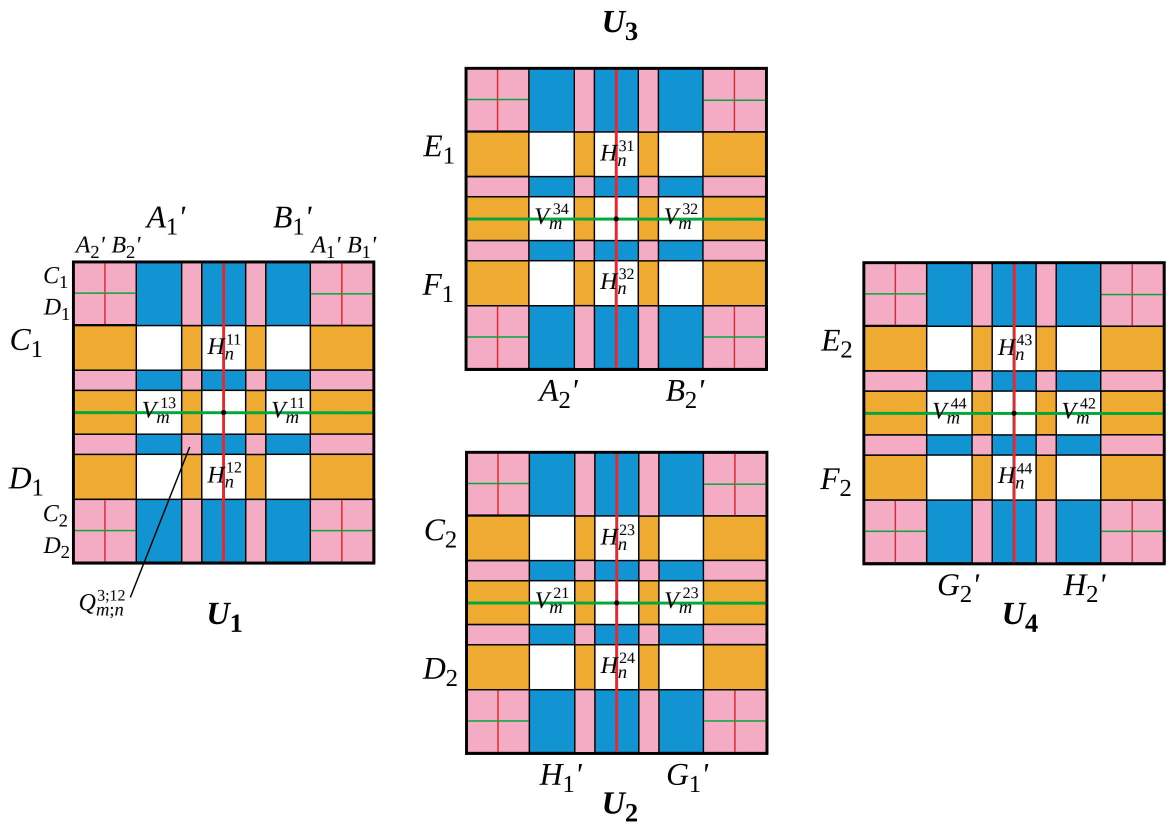
## Generalized Conley-Moser Conditions

- Proved $P$ satisfies **Generalized** Conley-Moser conditions:
  - **Strip** condition: it maps "horizontal strips" $H_n^{ij}$ to "vertical strips" $V_n^{ji}$.
  - **Hyperbolicity** condition: it has uniform contraction in horizontal direction and expansion in vertical direction.



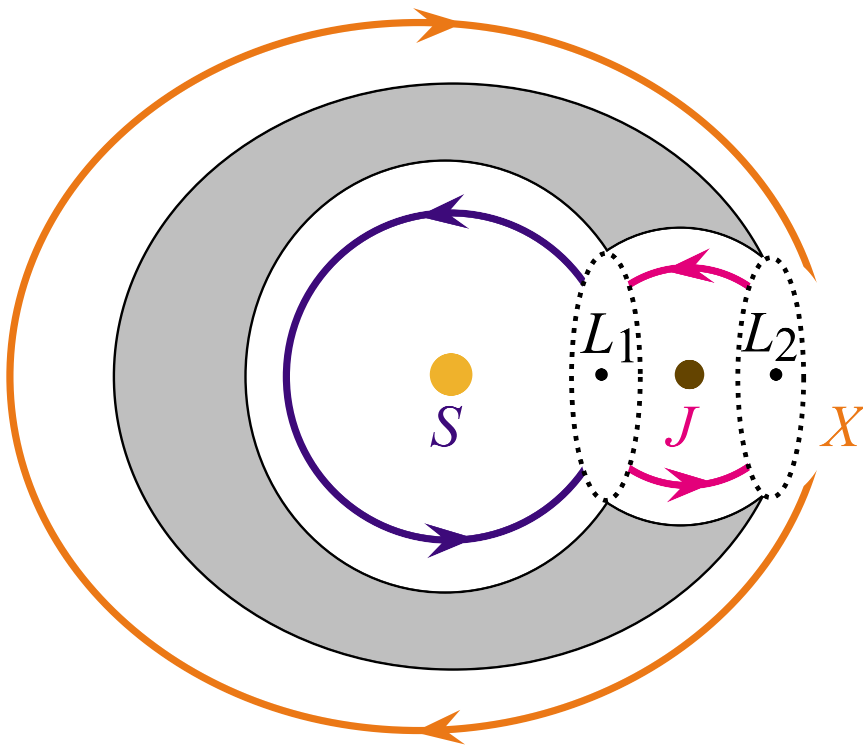
## Generalized Conley-Moser Conditions

- Shown are invariant set $\Lambda^1$ under **first iteration**.
- Since $P$ satisfies Generalized Conley-Moser Conditions, this process can be repeated **ad infinitum**.
- What remains is **invariant set** of points $\Lambda$ which are in 1-to-1 corr. with set of bi-infinite **sequences** $(\ldots, u_i, m; u_j, n, u_k, \ldots)$.



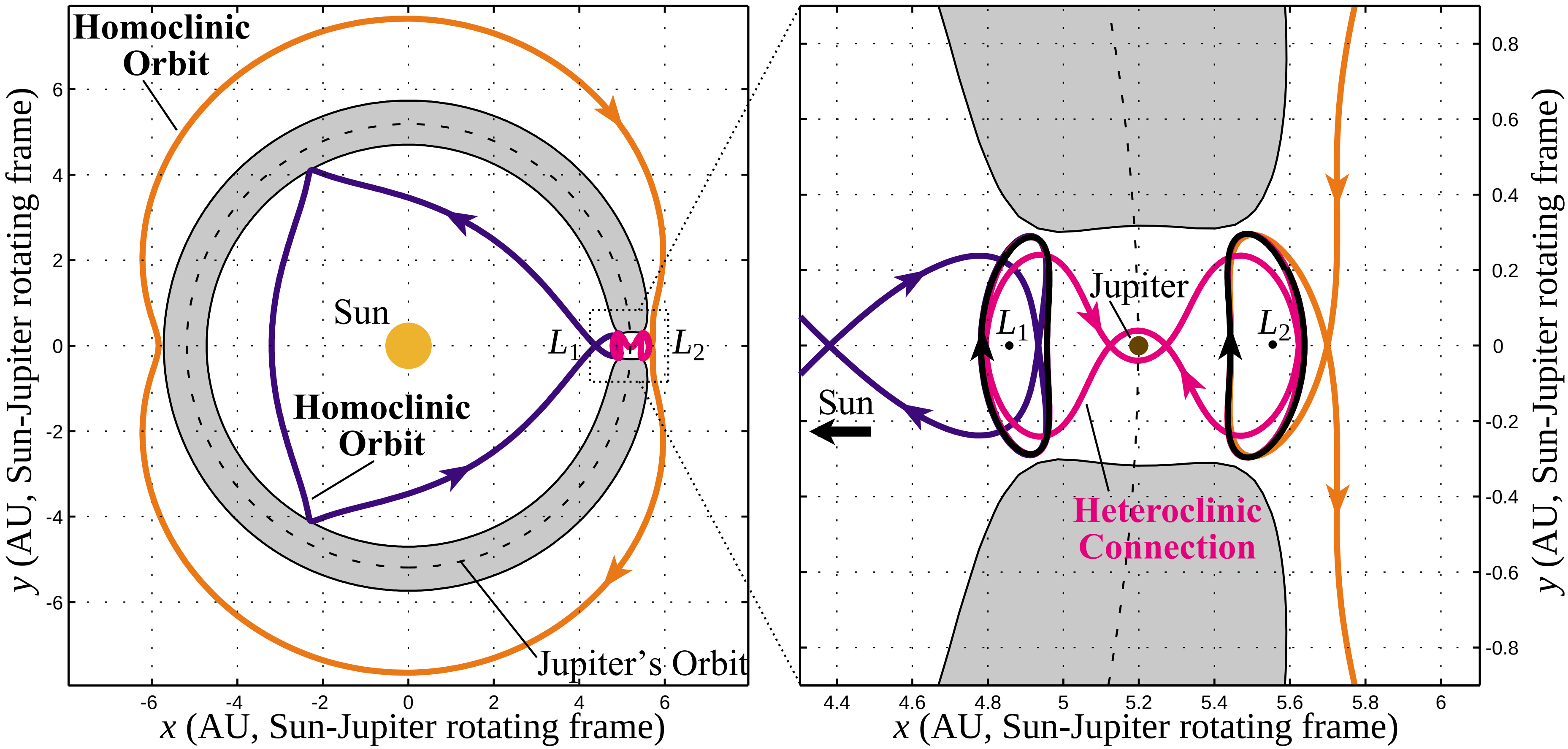
## Global Orbit Structure: Main Theorem

- ***Main Theorem***: For any admissible **itinerary**, e.g., $(\ldots, \mathbf{X}, \mathbf{1}, \mathbf{J}, \mathbf{0}; \mathbf{S}, \mathbf{1}, \mathbf{J}, \mathbf{2}, \mathbf{X}, \ldots)$, there exists an orbit whose **whereabouts** matches this **itinerary**.
- Can even specify **number of revolutions** the comet makes around Sun & Jupiter.



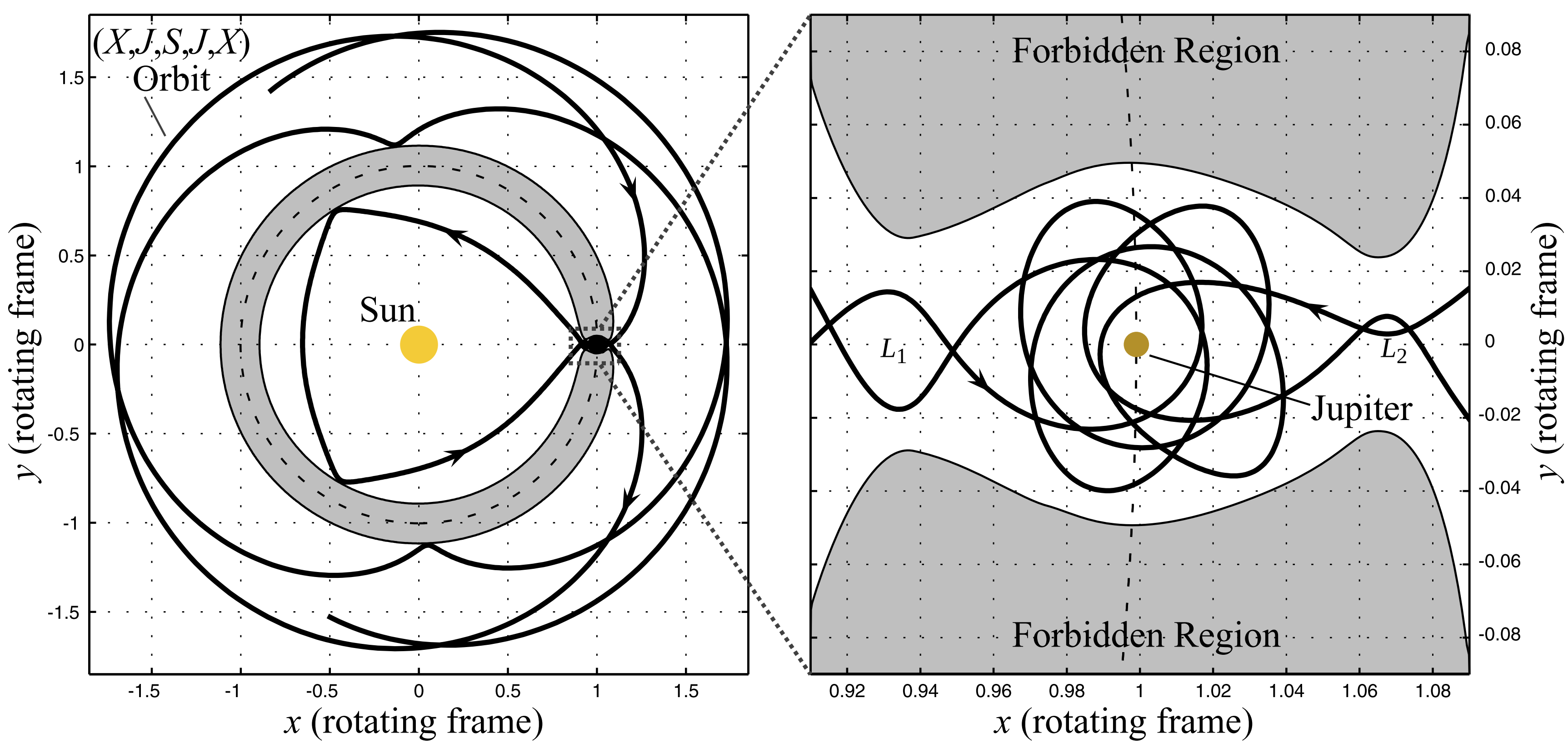
## Global Orbit Structure: Dynamical Channels

- Found a large class of **orbits** near homo/heteroclinic ***chain***.
- Comet can follow these ***channels*** in rapid transition.

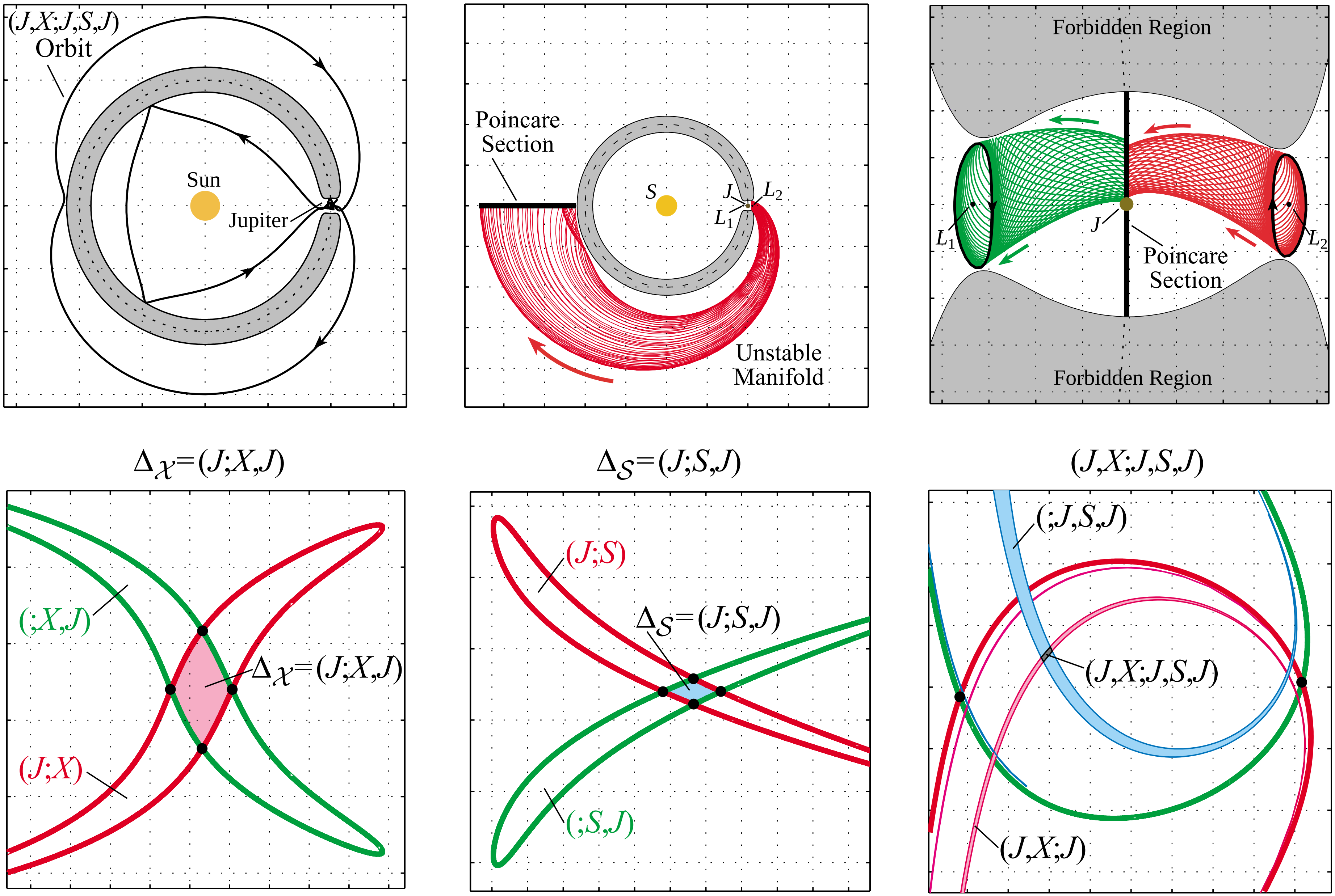


## Construction of Orbits with Prescribed Itinerary

- Using the proof of **Main Theorem** as the guide, we develop procedure to construct orbit with **prescribed itinerary**.
- Example: An orbit with itinerary $(\mathbf{X}, \mathbf{J}; \mathbf{S}, \mathbf{J}, \mathbf{X})$.
- **Petit Grand Tour** of Jovian moons & **Shoot the Moon**.



# Construction of (J, X; J, S, J) Orbits




**Exterior Region** **Interior Region** **Jupiter Region**