Dynamical Systems and Space Mission Design

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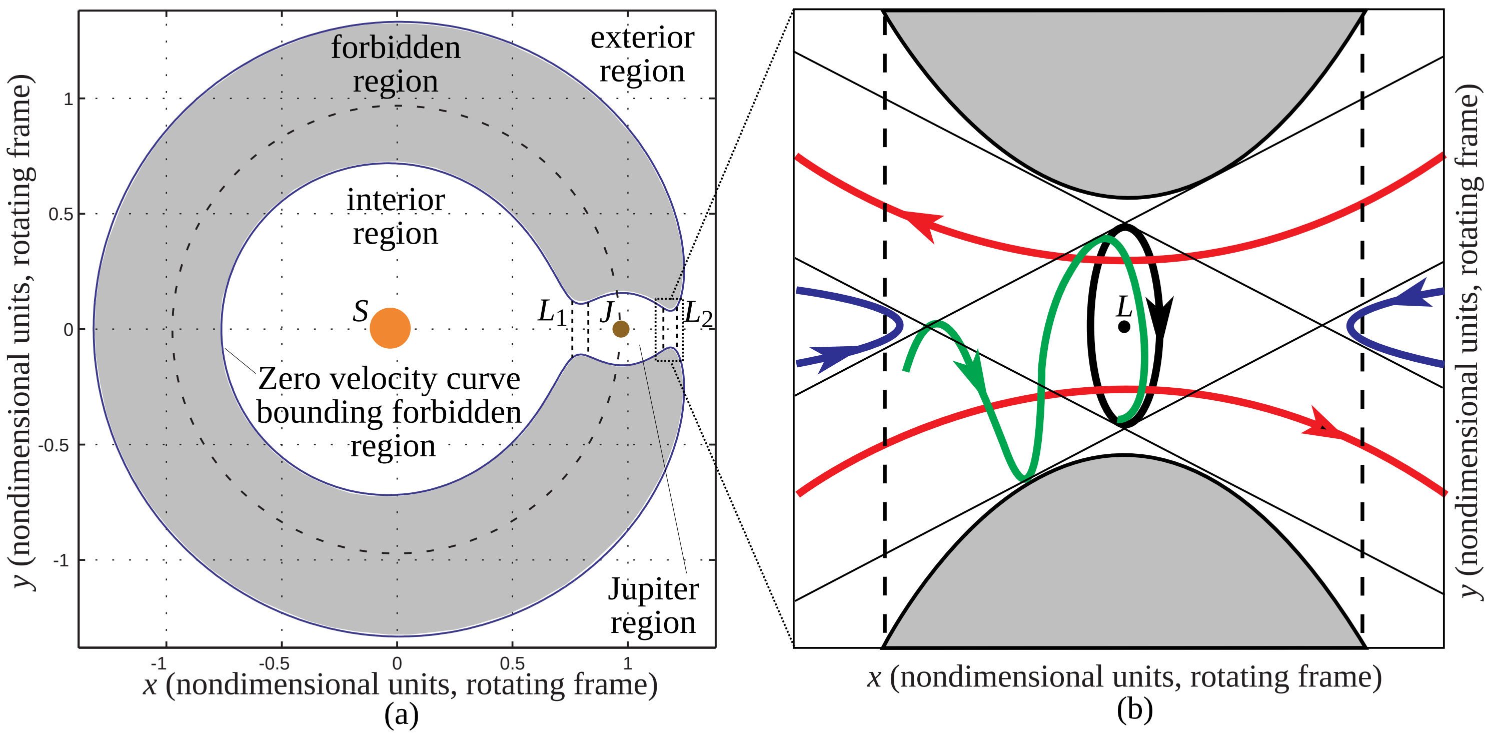
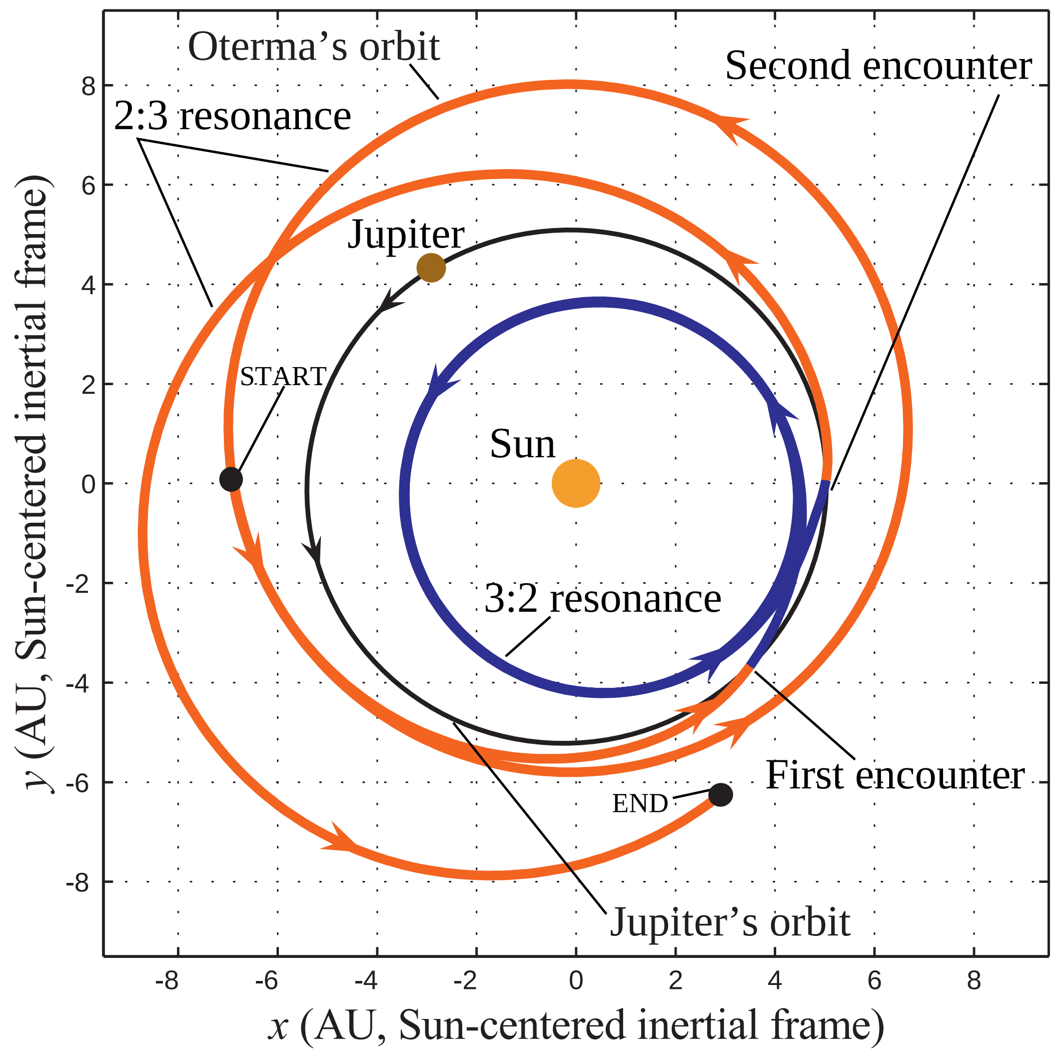
The Flow near L_1 and L_2: Outline

- Outline of Lecture 2B:
  - Equilibrium Regions near L_1 and L_2
  - Four Types of Orbits.
  - Invariant Manifold as Separatrix.
  - Flow Mappings in Equilibrium Region.



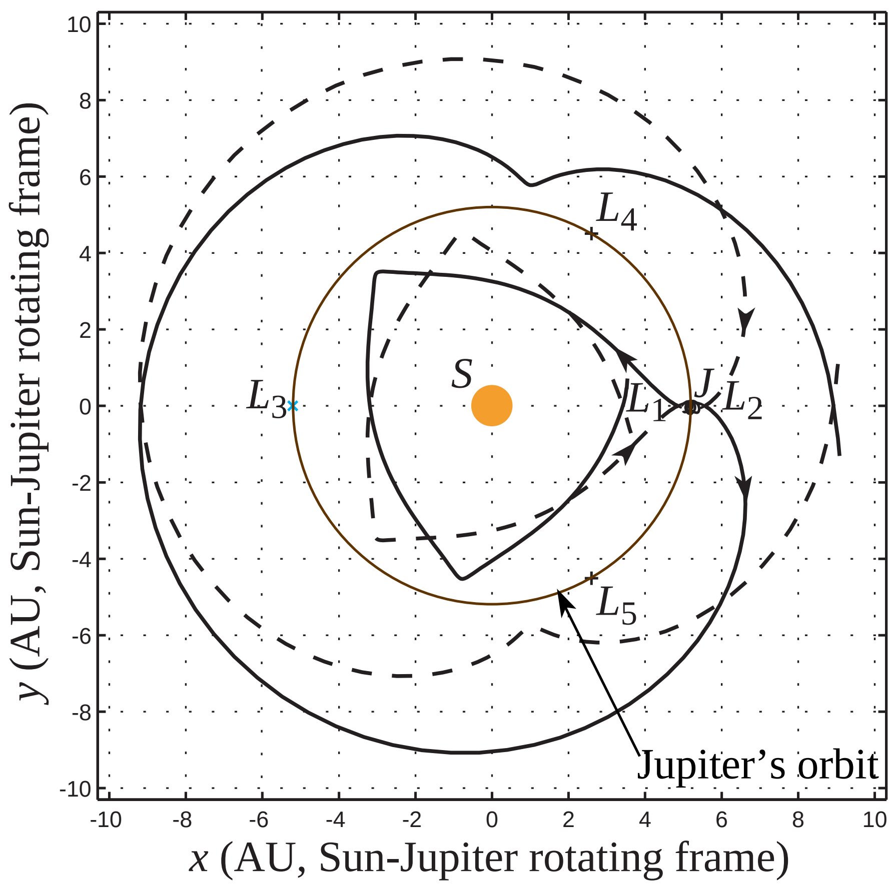
Jupiter Comets

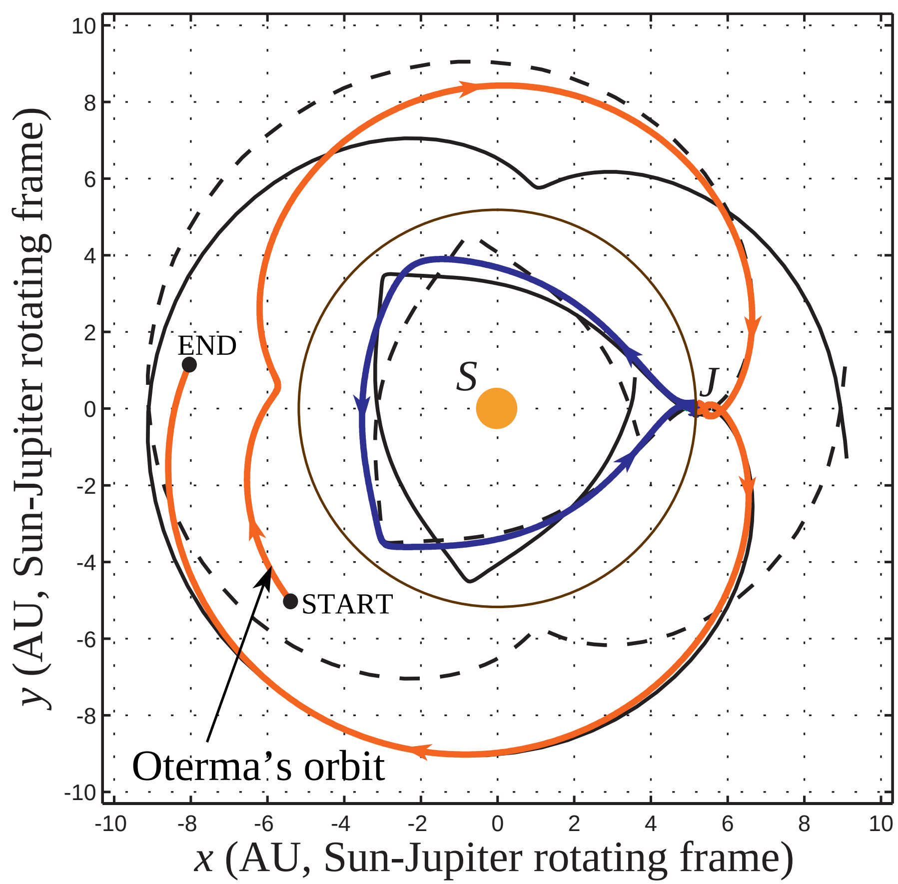
- **Rapid transition** from outside to inside Jupiter's orbit.
- **Captured temporarily** by Jupiter during transition.
- Exterior (**2:3 resonance**). Interior (**3:2 resonance**).



The Flow near L_1 and L_2: Overview

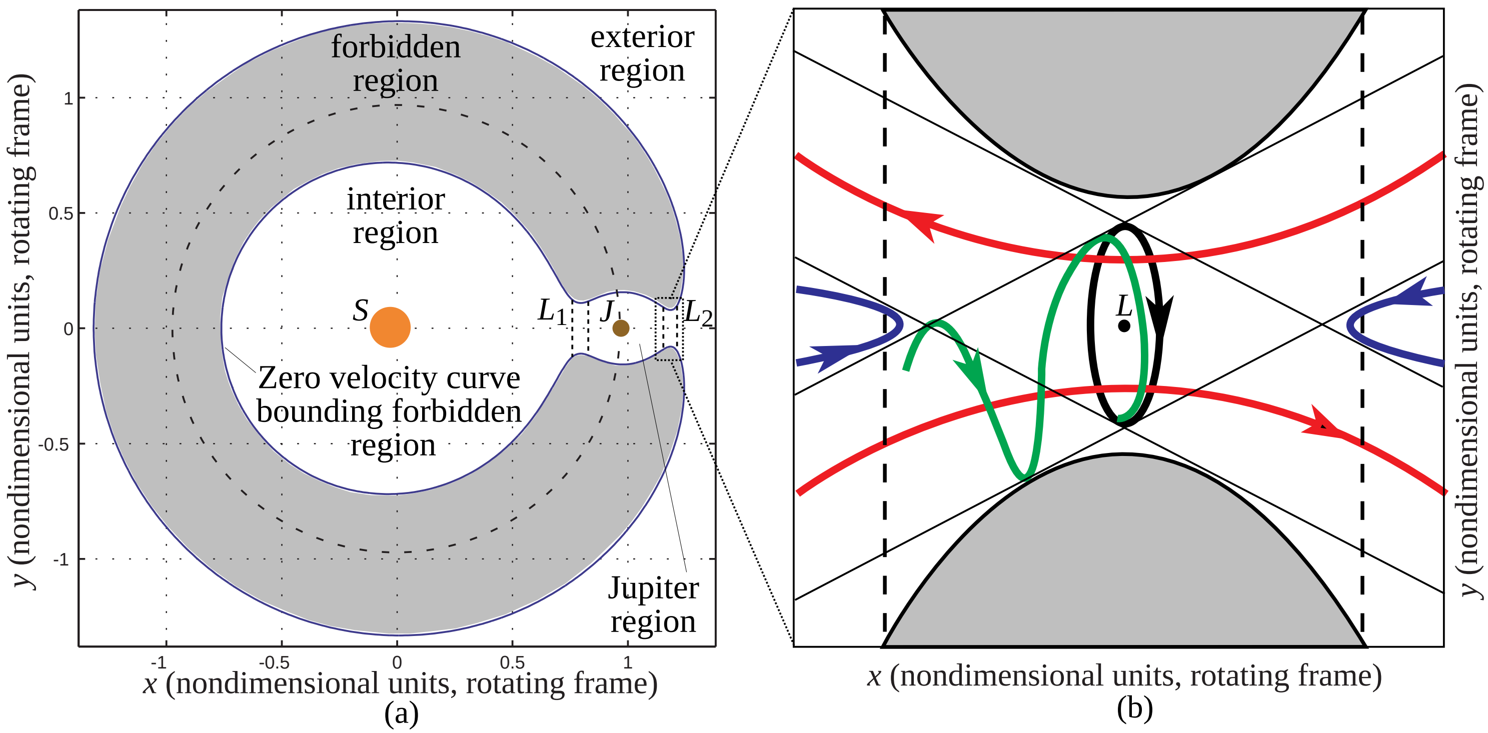
- Lo and Ross used PCR3BP as model and noticed that the comets follow closely the invariant manifolds of L_1 and L_2.
- Near L_1 and L_2 are where Jupiter comets make **resonance transition**.





The Flow near L_1 and L_2: Overview

- For energy value just above that of L_2,
  - **Hill's region** contains a "**neck**" about L_1 and L_2.
  - Comets are **energetically permitted** to make transition through these equilibrium regions.
- [Conley] The flow has **4 types** of orbits: **periodic**, **asymptotic**, **transit** and **non-transit** orbits.



The Flow near L_1 and L_2: Linearization

- [Moser] All the qualitative results of the linearized equations carry over to the full nonlinear equations.
- Recall equations of PCR3BP:

$$\begin{aligned} \dot{x} &= v_x, & \dot{v}_x &= 2v_y + \Omega_x, \\ \dot{y} &= v_y, & \dot{v}_y &= -2v_x + \Omega_y. \end{aligned} \tag{1}$$

- After linearization,

$$\begin{aligned} \dot{x} &= v_x, & \dot{v}_x &= 2v_y + ax, \\ \dot{y} &= v_y, & \dot{v}_y &= -2v_x - by. \end{aligned} \tag{2}$$

- Eigenvalues have the form $\pm\lambda$ and $\pm i\nu$.
- Corresponding eigenvectors are

$$\begin{aligned} u_1 &= (1, -\sigma, \lambda, -\lambda\sigma), \\ u_2 &= (1, \sigma, -\lambda, -\lambda\sigma), \\ w_1 &= (1, -i\tau, i\nu, \nu\tau), \\ w_2 &= (1, i\tau, -i\nu, \nu\tau). \end{aligned}$$

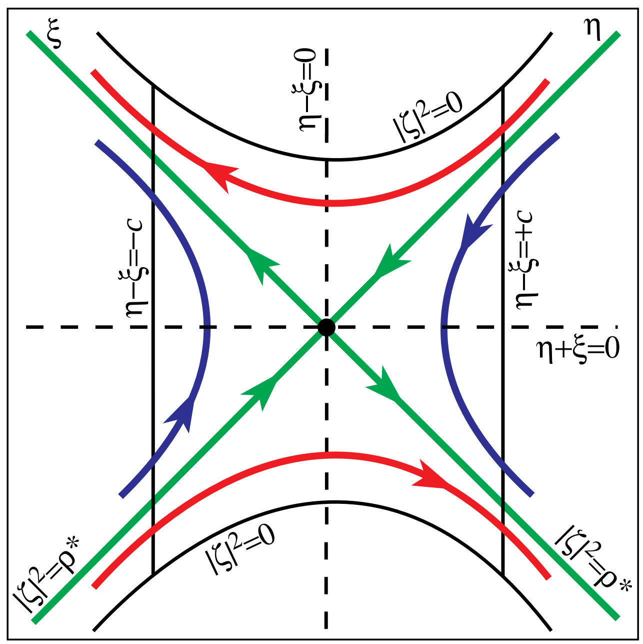
The Flow near L_1 and L_2: Linearization

- After **linearization** & making **eigenvectors** as new coordinate axes, equations assume simple form

$$\dot{\xi} = \lambda\xi, \qquad \dot{\eta} = -\lambda\eta, \qquad \dot{\zeta}_1 = \nu\zeta_2, \qquad \dot{\zeta}_2 = -\nu\zeta_1,$$

with **energy function** $E_l = \lambda\eta\xi + \frac{\nu}{2}(\zeta_1^2 + \zeta_2^2)$.

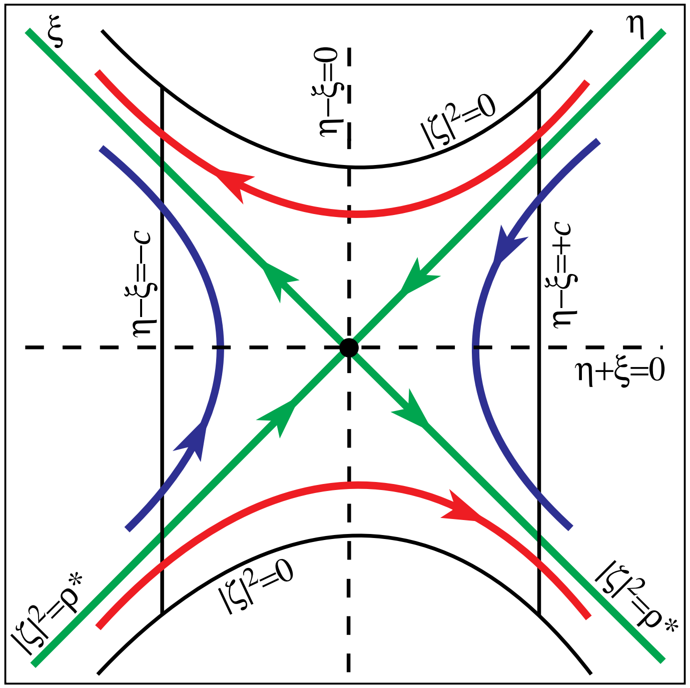
- The **flow** near L_1, L_2 have the form of **saddle**×**center**.



Flow near L_1 and L_2: Equilibrium Region $\mathcal{R} \simeq S^2 \times I$

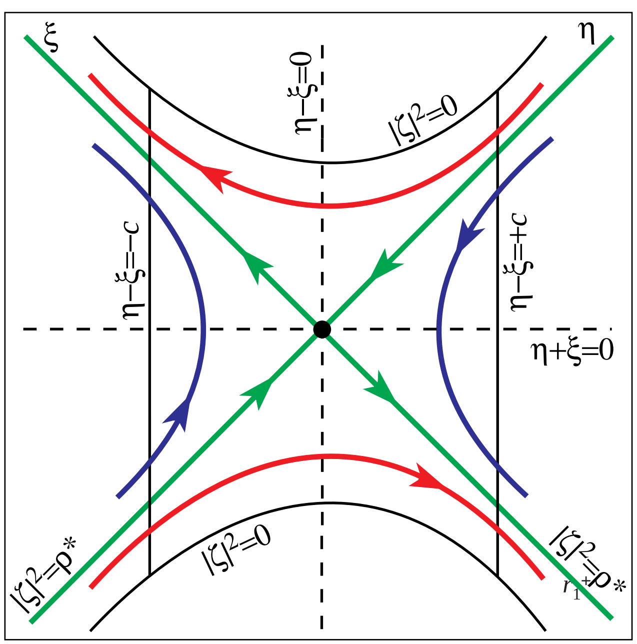
- Recall that **energy function** $E_l = \lambda\eta\xi + \frac{\nu}{2}(\zeta_1^2 + \zeta_2^2)$.
- **Equilibrium region** $\mathcal{R}$ on **3D energy manifold** is homeomorphic to $S^2 \times I$.
- Because for each fixed value of $\eta - \xi$, $E_l = \mathcal{E}$ is a **2-sphere**

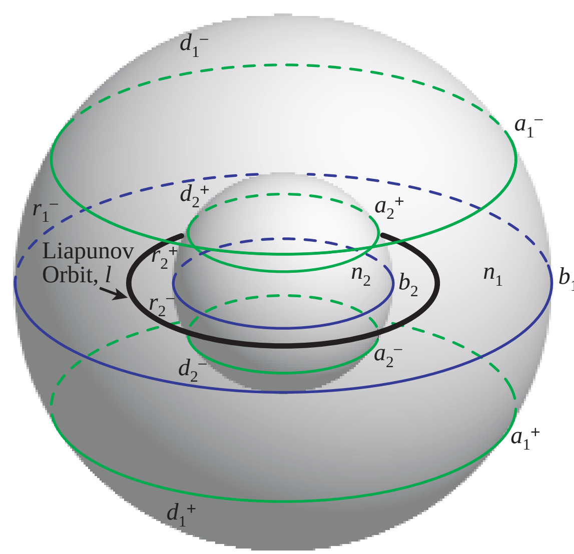
$$\frac{\lambda}{4}(\eta + \xi)^2 + \frac{\nu}{2}(\zeta_1^2 + \zeta_2^2) = \mathcal{E} + \frac{\lambda}{4}(\eta - \xi)^2.$$



Flow near L_1 and L_2: Equilibrium Region $\mathcal{R} \simeq S^2 \times I$

- Recall that **energy function** $E_l = \lambda\eta\xi + \frac{\nu}{2}(\zeta_1^2 + \zeta_2^2)$.
  - Each **point** in (η, ξ)-plane corr. to a **circle** S^1 in $\mathcal{R}$ with radius $|\zeta| = \sqrt{\frac{2}{\nu}(\mathcal{E} - \lambda\eta\xi)}$,
  - with **point** on the **bounding hyperbola** corr. to **point-circle**.
- Thus, **line segment** $(\eta - \xi = \pm c)$ corr. to S^2 in $\mathcal{R}$ because each corr. to $S^1 \times I$ with **2 ends** pinched to a **point**.





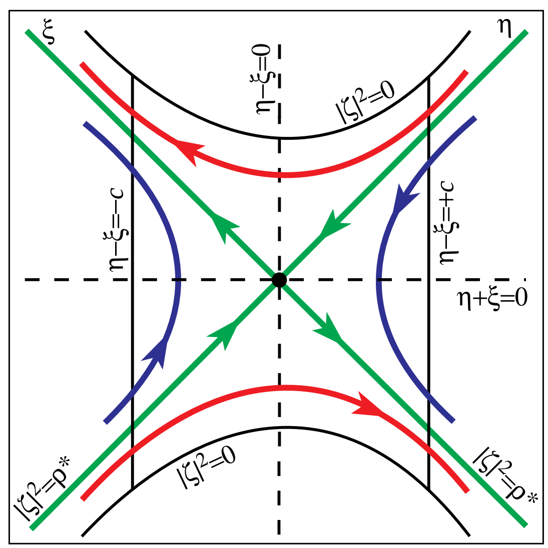
Flow near L_1 and L_2: 4 Types of Orbits

- The **flow** $(\eta(t), \xi(t), \zeta_1(t), \zeta_2(t))$ is given by

$$\eta(t) = \eta^0 e^{-\lambda t}, \qquad \xi(t) = \xi^0 e^{\lambda t},$$
$$\zeta(t) = \zeta_1(t) + i\zeta_2(t) = \zeta^0 e^{-i\nu t},$$

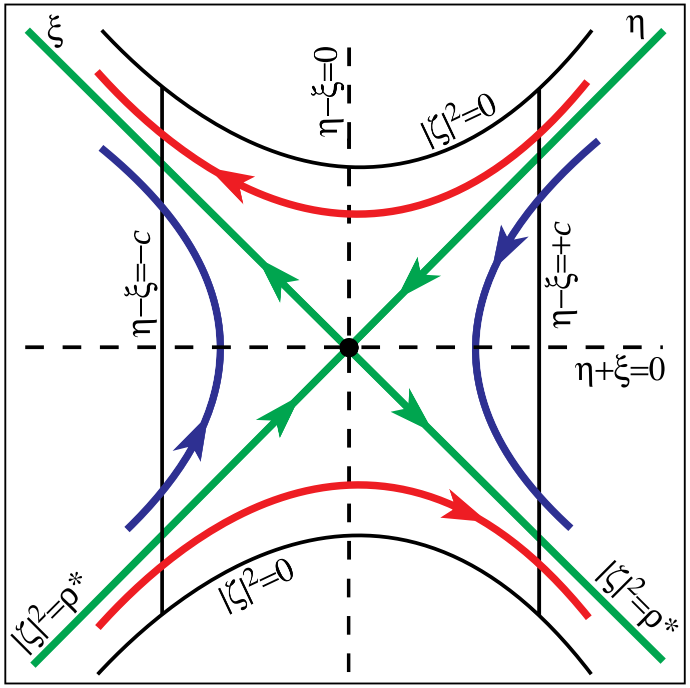
with 2 additional **integrals** $\eta\xi (= \eta^0\xi^0)$, $|\zeta|^2 = \zeta_1^2 + \zeta_2^2 (= |\zeta^0|^2)$.

- 9 classes of orbits grouped into **4 types**:
  - **black** point $\eta = \xi = 0$ corr. to a **periodic** orbit.
  - 4 half **green** segments of axis $\eta\xi = 0$ corr. to 4 cylinders of orbits **asymptotic** to this periodic orbit.



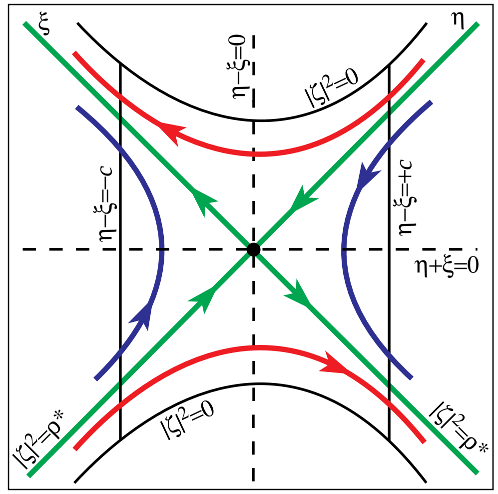
Flow near L_1 and L_2: 4 Types of Orbits

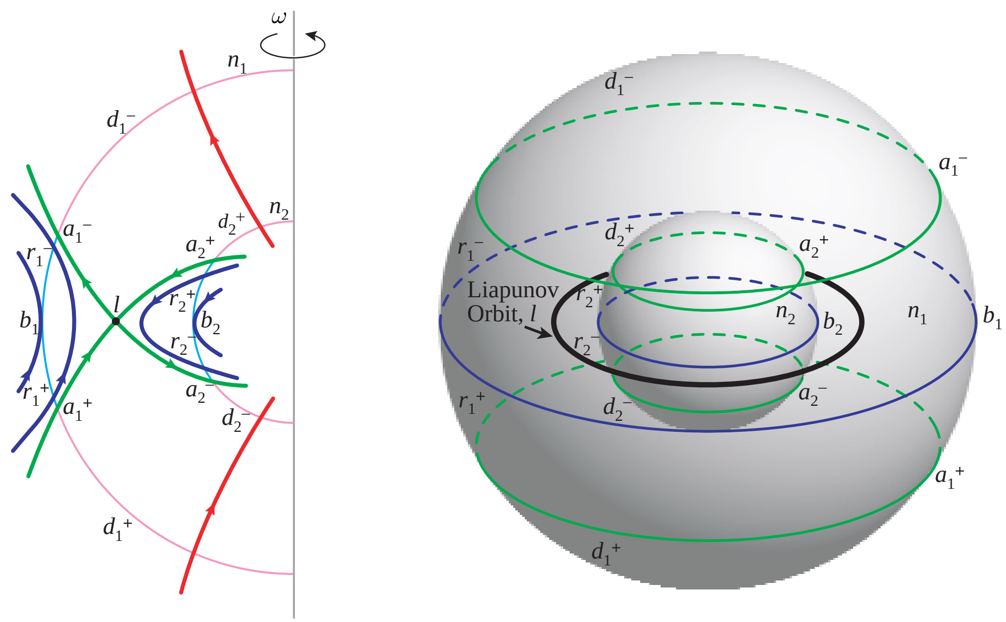
- 9 classes of orbits grouped into **4 types**: Shown are
  - 2 **red** hyperbolic segments $\eta\xi$ = constant > 0 corr. to 2 cylinders of **transit orbits** which transit from 1 bounding sphere to the other.
  - 2 **blue** hyperbolic segments $\eta\xi$ = constant < 0 corr. to 2 cylinders of **non-transit orbits** each of which runs from 1 hemisphere to the other on same bounding sphere.



McGehee Representation: Equilbrium Region $\mathcal{R}$

- [McGehee] To visualize region $\mathcal{R} \simeq S^2 \times I$.

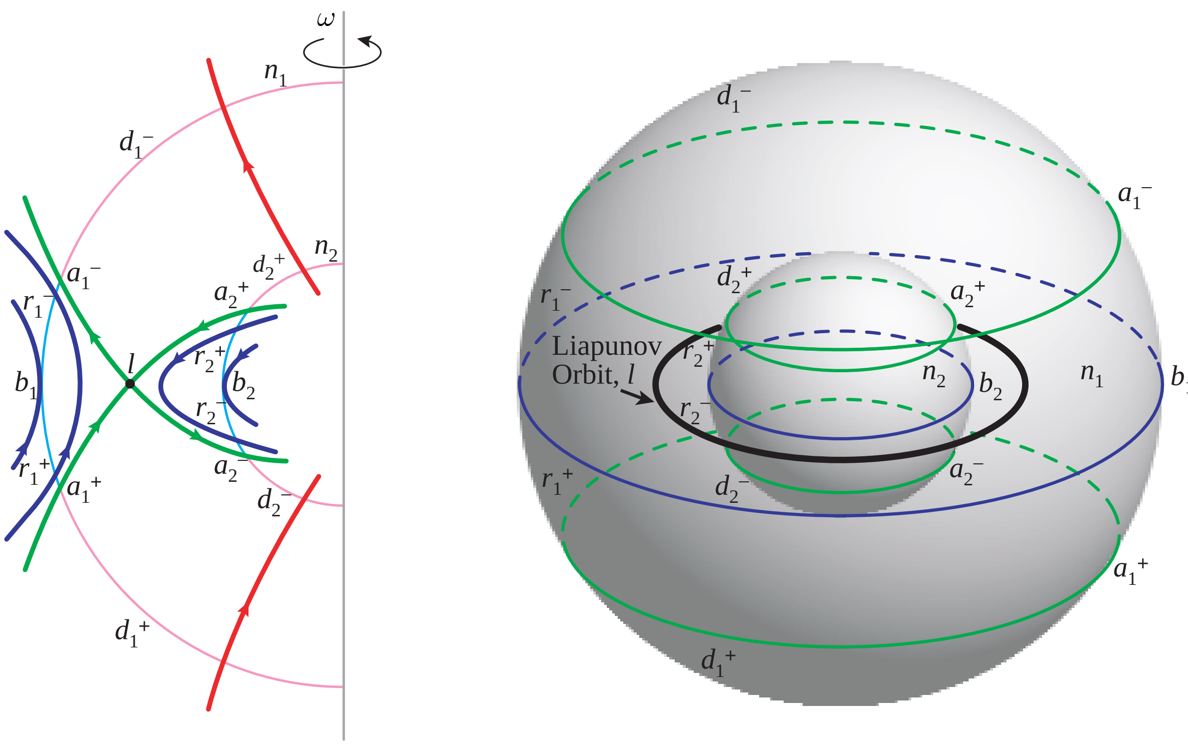




(a) (b)

McGehee Representation: Equilibrium Region $\mathcal{R}$

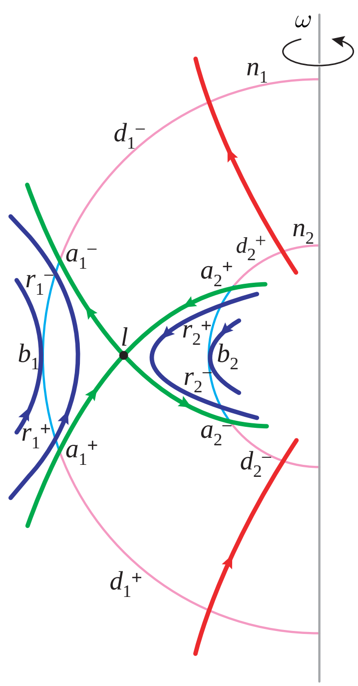
- **4 types** of orbits in equilibrium region $\mathcal{R}$:
  - **Black** circle l is the unstable **periodic** orbit.
  - 4 cylinders of **asymptotic** orbits form pieces of **stable** and **unstable manifolds**.
    They intersect the bounding spheres at **asymptotic circles**.



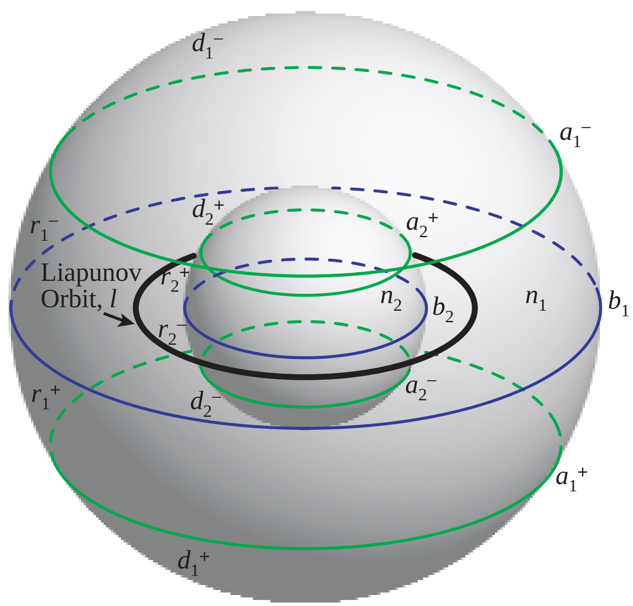
(a) (b)

McGehee Representation: Equilibrium Region $\mathcal{R}$

- **Asymptotic circles** divide bounding sphere into spherical **caps** and spherical **zones**.
- **4 types** of orbits in equilibrium region $\mathcal{R}$:
  - **Transit** orbits entering $\mathcal{R}$ through a **cap** on a bounding sphere will leave through a **cap** on another bounding sphere.
  - **Non-transit** orbits entering $\mathcal{R}$ through a spherical **zone** will leave through another **zone** of the same bounding sphere.



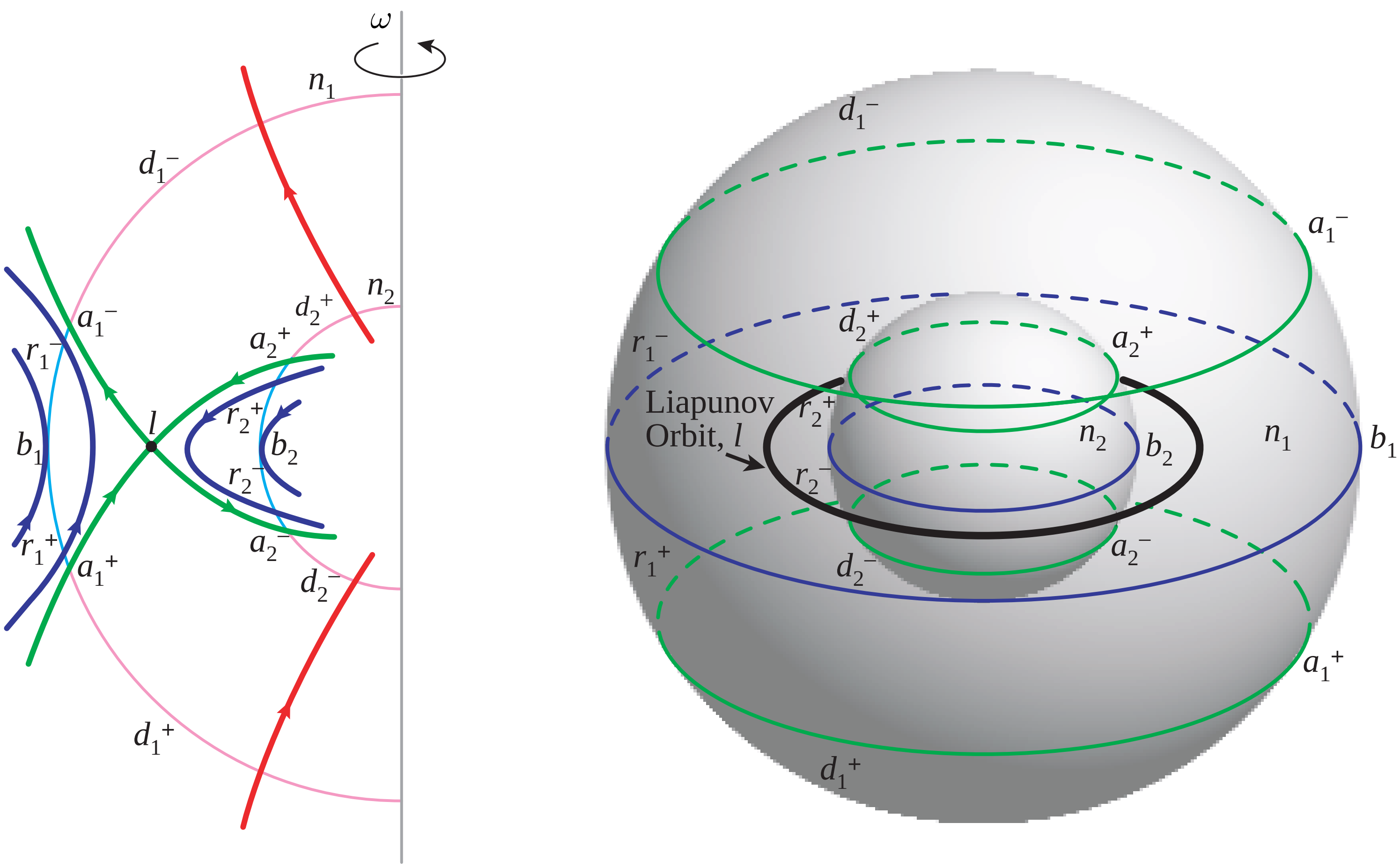
(a)



(b)

McGehee Representation: Separatrix

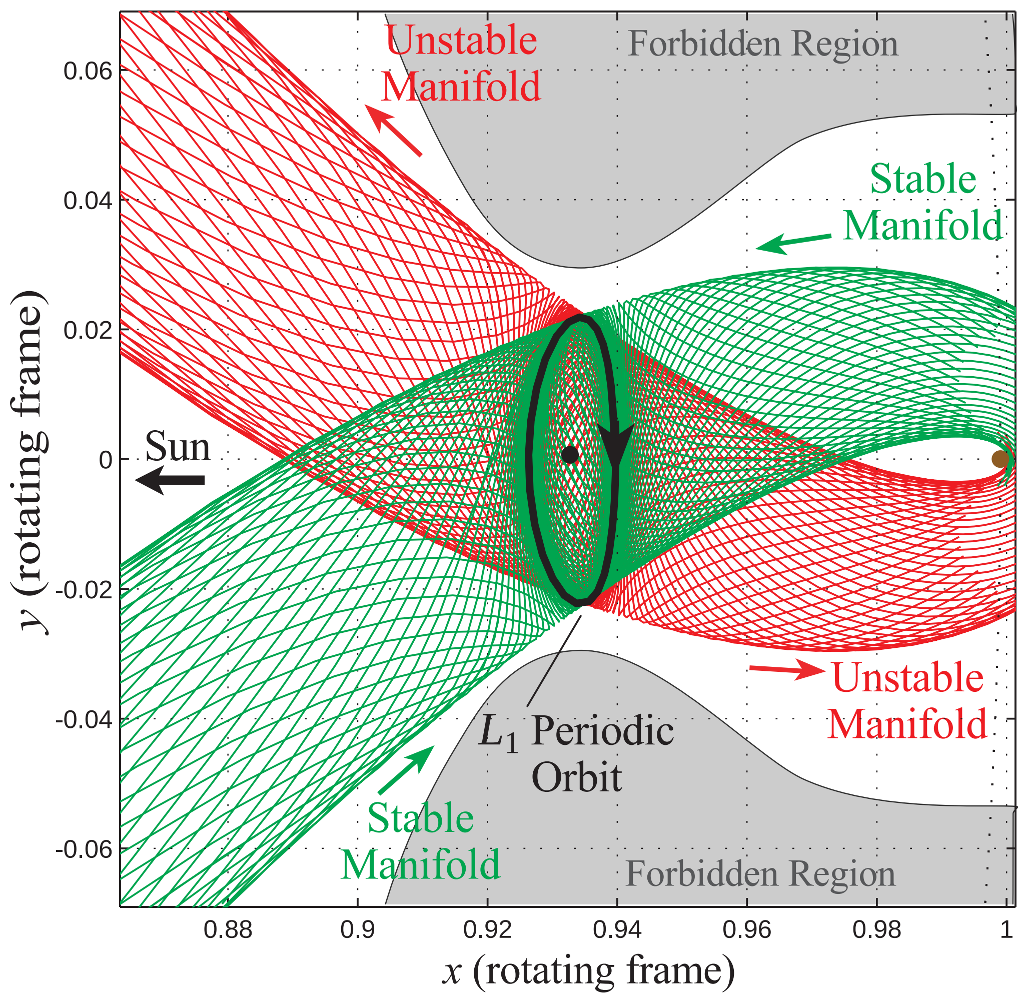
- **Invariant manifold tubes** act as **separatrices** for the flow in $\mathcal{R}$:
  - Those inside the tubes are **transit** orbits.
  - Those outside of the tubes are **non-transit** orbits.



(a) (b)

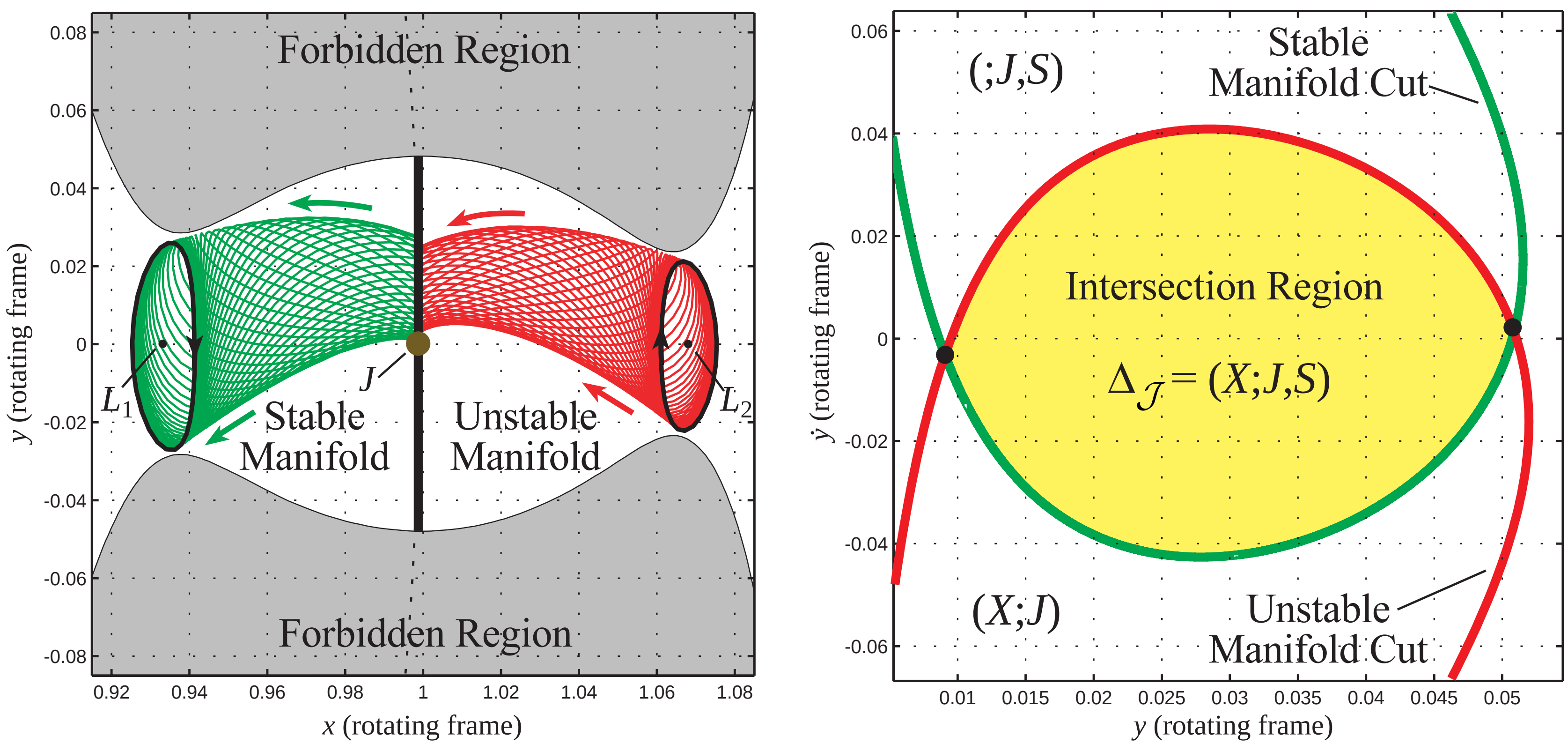
McGehee Representation: Separatrix

- **Stable** and **unstable manifold tubes** act as **separatrices** for the flow in $\mathcal{R}$:
  - Those inside the tubes are **transit** orbits.
  - Those outside of the tubes are **non-transit** orbits.



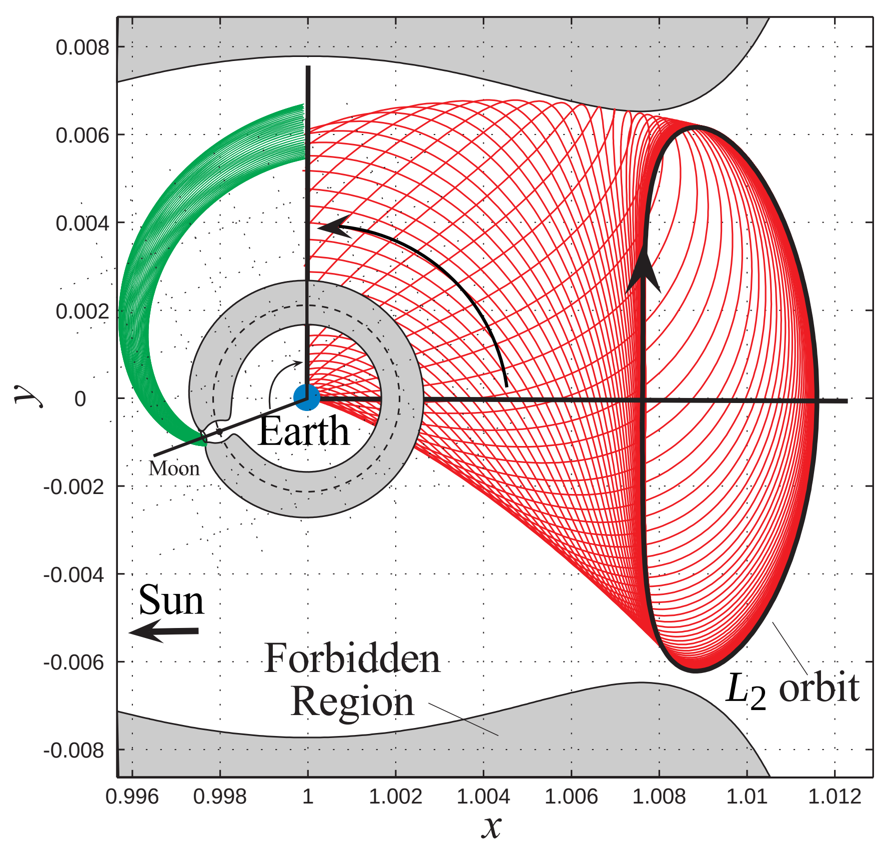
Applications: Jupiter Comets

- By linking **invariant manifold tubes**, we found **dynamical channels** for a fast transport mechanism between **exterior** and **interior** Hill's regions.
- Jupiter comets can follow these **channels** in rapid transition between **exterior** and **interior** region passing through **Jupiter** region.



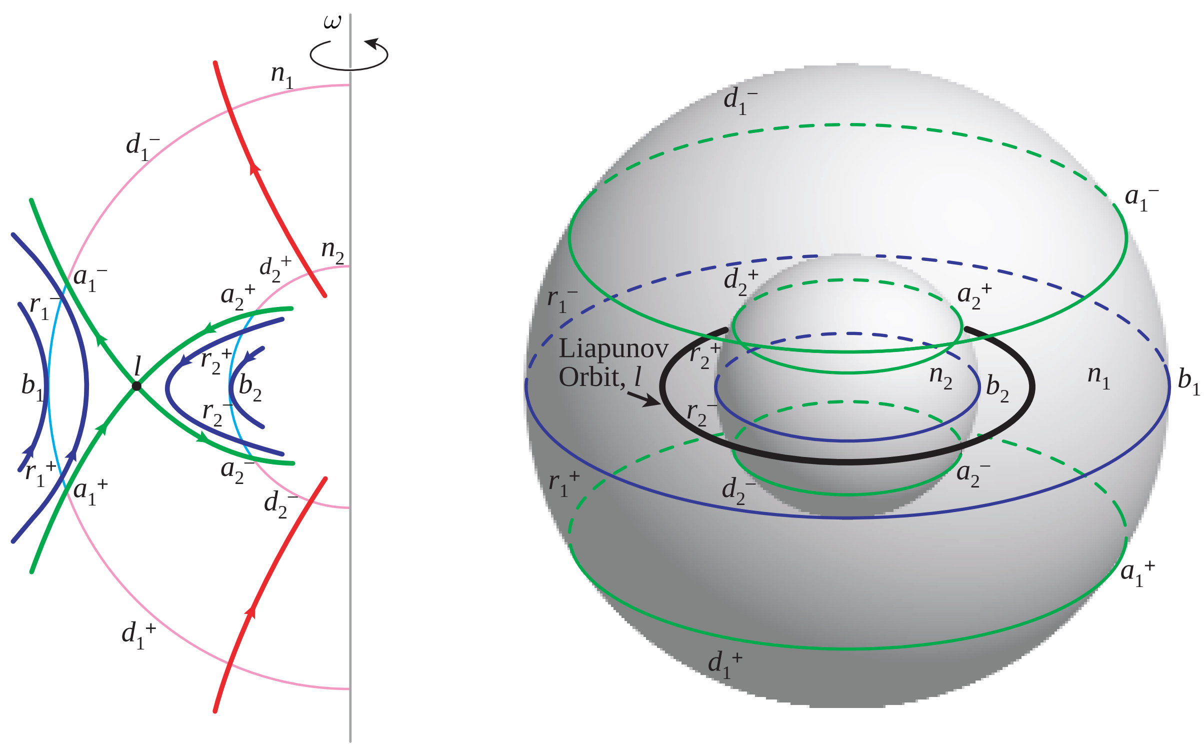
Applications: Shoot the Moon

- **Stable manifold tube** provide a **temporary capture** mechanism by the second primary.
- **Stable manifold tube** of a periodic orbit around L_2 guides spacecraft towards a **ballistic capture** by the Moon.
- By saving fuel for this lunar ballistic capture leg, the design uses **less** fuel than a Earth-to-Moon Hohmann transfer.



Flow Mappings in $\mathcal{R}$: 4 Flow Mappings

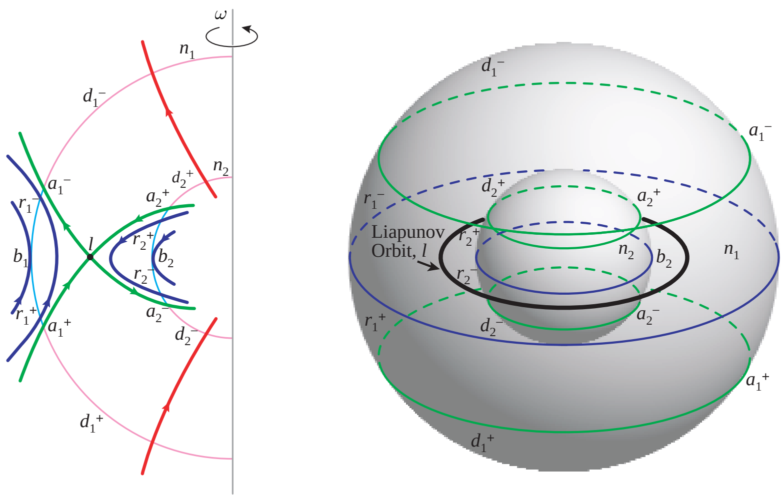
- The **flow** in $\mathcal{R}$ defines 4 **mappings**:
  - **2** between pairs of spherical **caps** of different bounding spheres.
  - **2** between pairs of spherical **zones** of same bounding sphere.
- They are Poincaré maps in $\mathcal{R}$ and they will be used later to build the global **Poincaré map**.



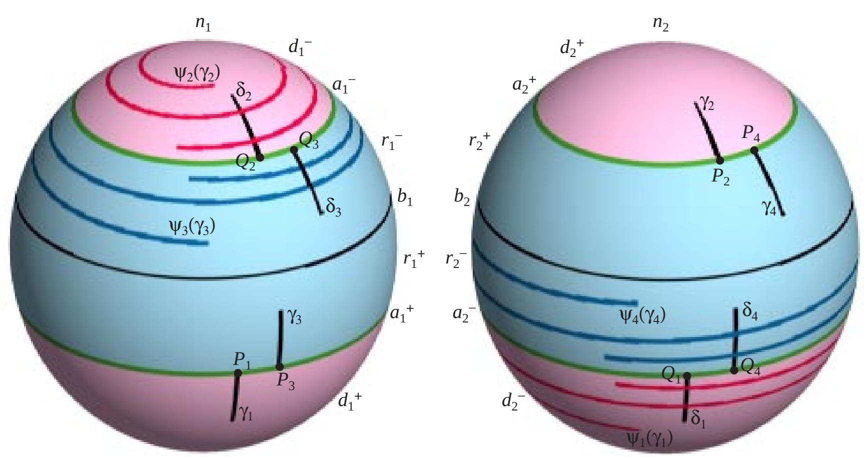
(a) (b)

Flow Mappings in $\mathcal{R}$: Infinite Twisting of the Map

- To visualize the infinite twisiting of the maps.

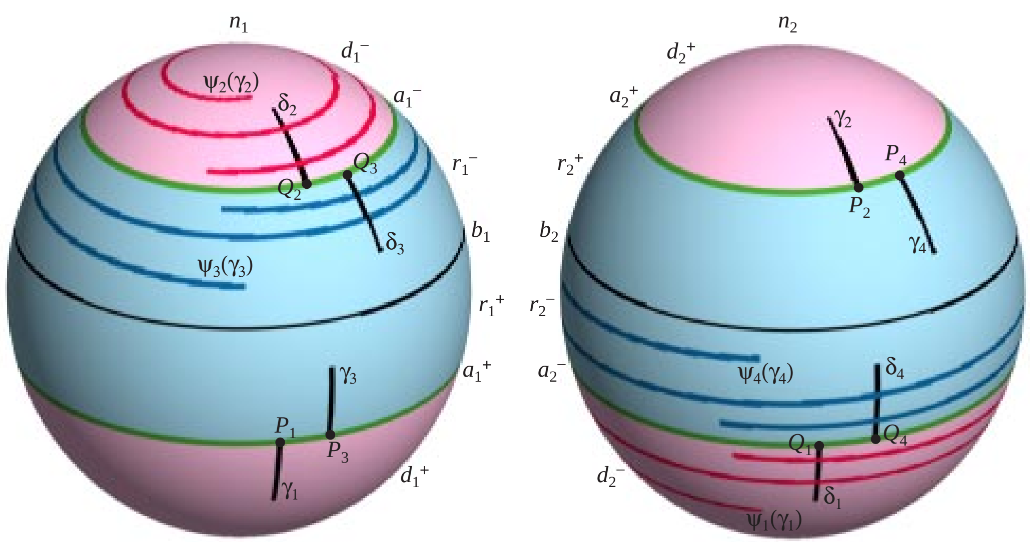


(a) (b)



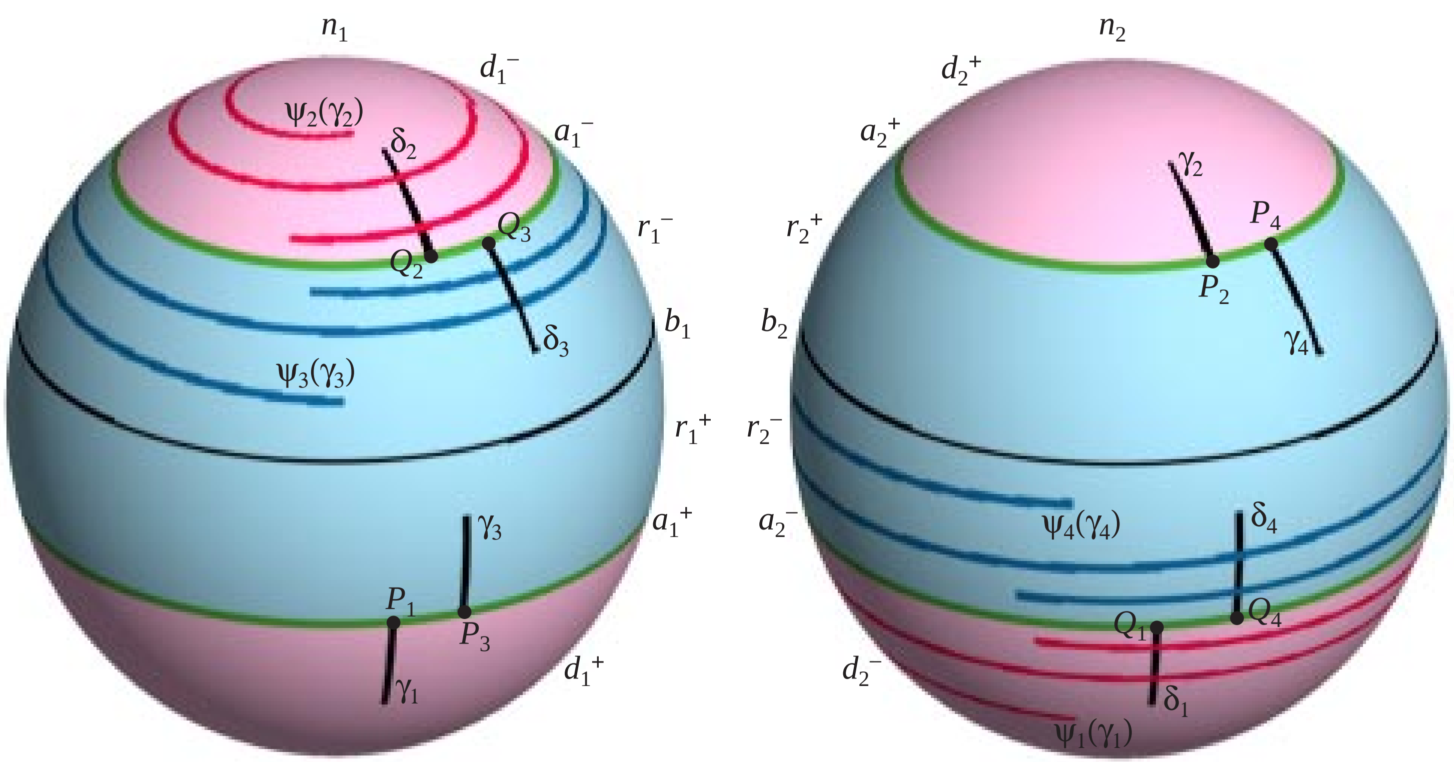
Flow Mappings in $\mathcal{R}$: Infinite Twisting of the Map

- From $\zeta(t) = \zeta^0 e^{-i\nu t}$, we obtain $\frac{d}{dt}(\arg \zeta) = -\nu$.
- Amount of **twisting** a point will undergo is proportional to the **time** required for its corr. trajectory to go from domain to range.
- This **time** approaches **infinity** as the flow approaches an **asymtotic circle**.
- Hence, amount of **twisiting** a point will undergo depends very sensitively on its **distance** from an **aymptotic circle**.



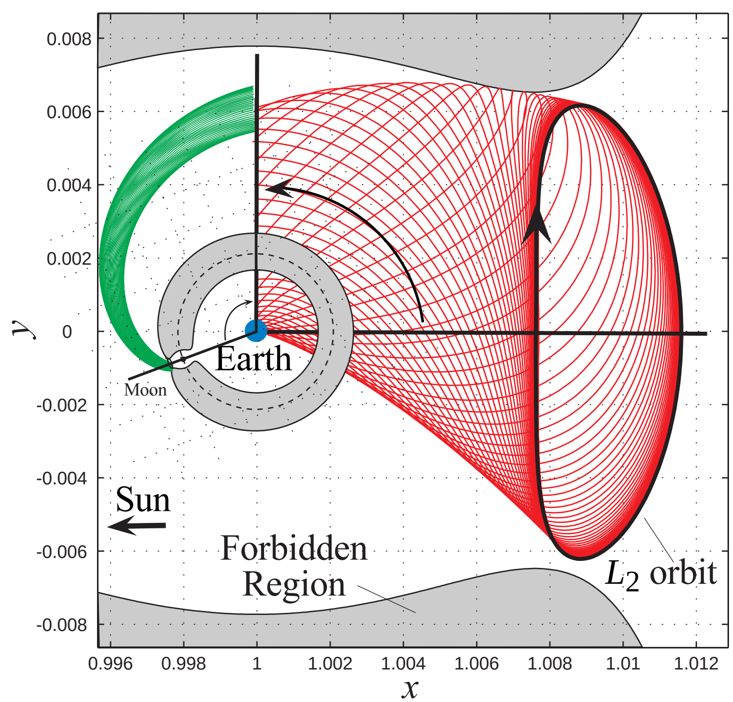
Flow Mappings in $\mathcal{R}$: Infinite Twisting of the Map

- **Images** $\psi_2(\gamma_2)$ of **abutting arc** γ_4 **spirals infinitely many times** around and towards **asymptotic circles** a_1^-.



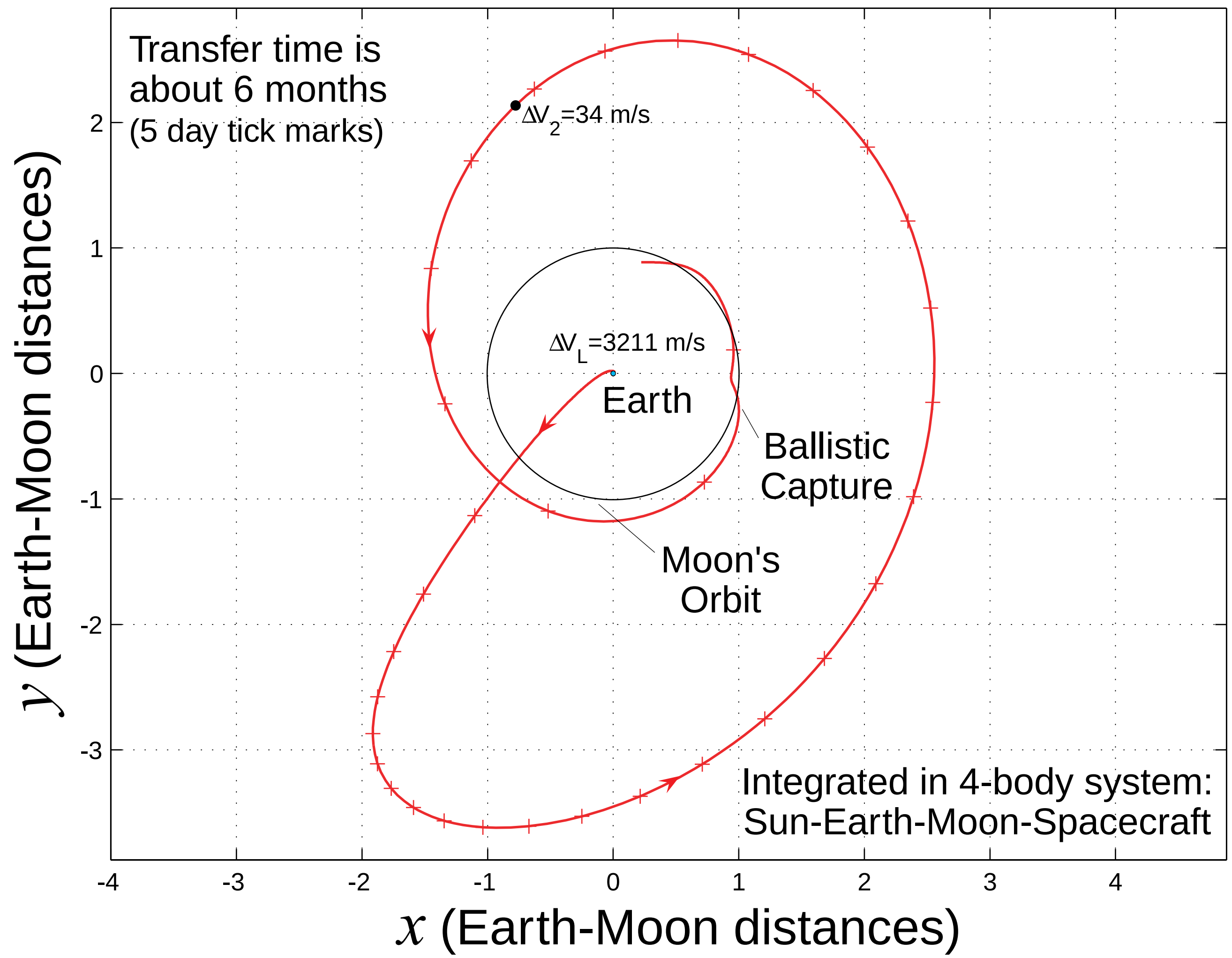
Applications: Shoot the Moon

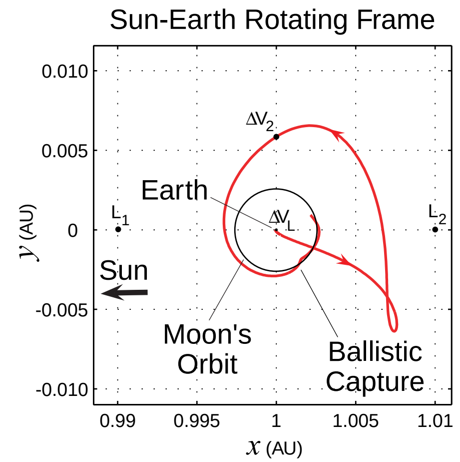
- Find position and velocity for a spacecraft such that
  - when integrating **forward**, SC will be guided by **Earth-Moon manifold** and get ballistically captured at the Moon;
  - when integrating **backward**, SC will hug **Sun-Earth manifolds** and return to the Earth.
- **Infinite twisting** of flow map is key in finding a **low energy transfer** in Sun-Earth leg of the design.

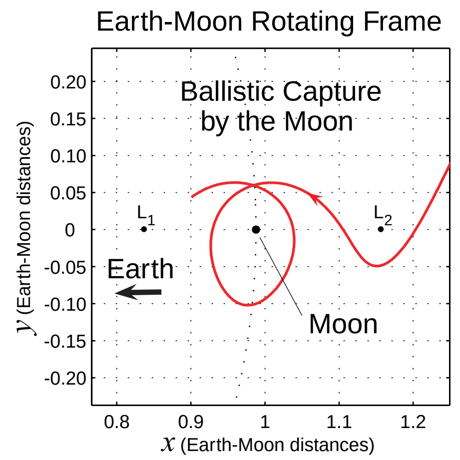


(a)

Ballistic Capture of a Spacecraft by the Moon from 200 km Altitude Earth Parking Orbit (Shown in an Inertial Frame)

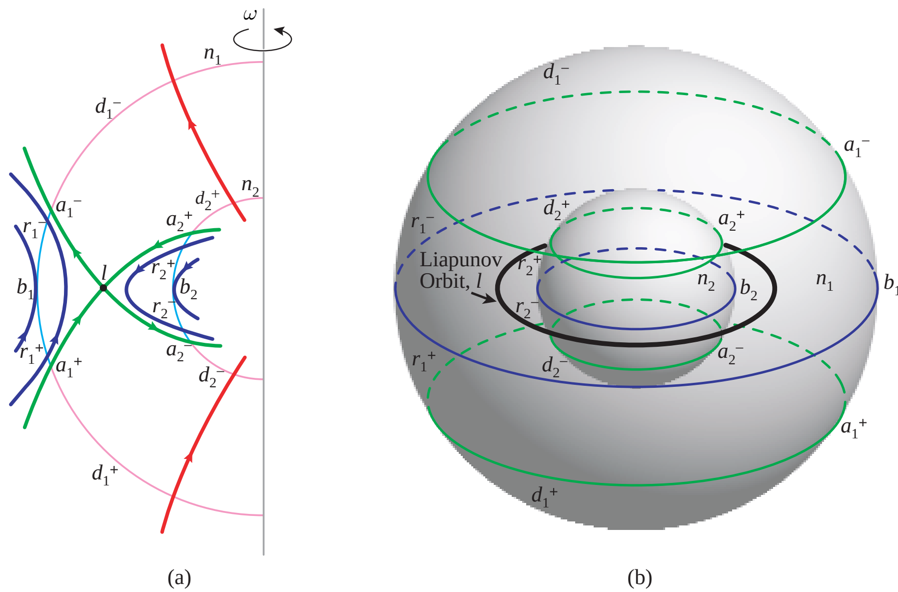


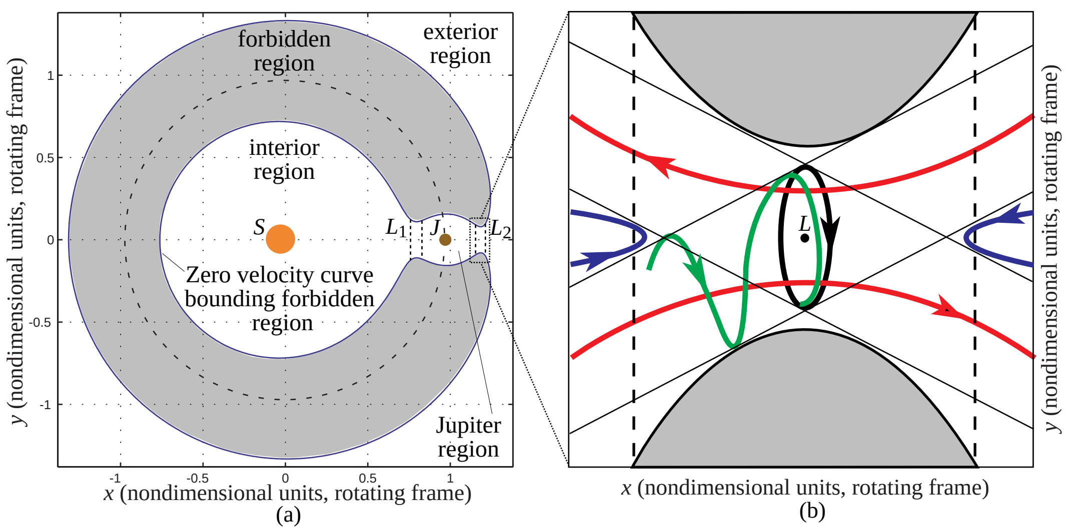




Appearance of Orbits in Position Space

- **Tilted projection** of $\mathcal{R}$ on (x, y)-plane.





Appearance of Orbits in Position Space

- **4 types** of orbits: Shown are
  - A **periodic** orbit.
  - A typical **aymptotic** orbit.
  - 2 **transit** orbits.
  - 2 **non-transit** orbits.