Dynamical Systems, and Space Mission Design

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Outline for Lecture 2A

- Equations of motion (using both Langrangian and Hamiltonian approaches).
- Energy and the Jacobi constant.
- Equilibrium points and their stability.
- Hill's region.

The PCR3BP

- Stands for:
  Planar Circular Restricted Three Body Problem.
- Describes the motion of a body moving in the gravitational field of two ***primaries*** that are moving in circles.

- The two primaries could be the Sun and Jupiter, the Sun and Earth, etc.
- Let μ be the ratio between the mass of the Earth and the mass of the Sun-Earth system,

$$\mu = \frac{m_J}{m_J + m_S},$$

- For the Sun-Jupiter system, $\mu = 9.537 \times 10^{-4}$.
- For the Sun-Earth system, $\mu = 3.03591 \times 10^{-6}$.
- The primaries rotate about their center of mass, with angular velocity normalized to 1.
- We will usually use a rotating coordinate system with origin at the primaries center of mass so that S and J are located at the points $(-\mu, 0)$ and $(1-\mu, 0)$.

When is the PCR3BP an appropriate starting point?

■ **Astronomical Phenomena.** For phenemena like resonance transition, we consider it an adequate starting model.

- Comets of interest are mostly ***heliocentric***, but their perturbation are dominated by ***Jupiter's gravitation***.
- Their motion is nearly in Jupiter's ***orbital plane***, and the small eccentricity of Jupiter's orbit (i.e., it is nearly circular) plays little role during resonant transition.
- In a more detailed study, and for verification that one can really believe the PCR3B model, one needs to to take into account the 3D effects and the perturbation from Saturn.

For Space Mission Design. For systems such as the ***Genesis Discovery Mission*** and ***Shoot the Moon***:

- For Genesis, the knowledge about heteroclinic behavior provided the necessary insight in searching for the desired solution.
- For Shoot the Moon, the study of PCR3BP provides a systematic method for the numerical construction of the trajectory.
- In both cases, the trajectores found in the simpler models provided the starting point for the differential correction process to produce the final trajectory.

Why use Rotating Frames?

Visual.

- It is quite common to see much more structure in carefully chosen ***rotating frames*** than in stationary ones, even in cases where there is no "obvious" preferred rotating frame. This is well illustrated by the ***DSP: double spherical pendulum***.
- Look at the accompanying movie clips on the **DSP**.
- In the Sun-Jupiter rotating frame, one can clearly observe that near L_1 and L_2 the comet ***Oterma*** makes a transition between the exterior and the interior region.
- Also, one can see how closely the orbit of ***Oterma*** follows the plots of invariant manifolds of L_1 and L_2 in the position space.

Analytical

- Analytically, in this preferred Sun-Jupiter rotating frame, the equations of motion of the comet is autonomous. It has an integral of motion and it has equilibrium points which allow us to bring in all the tools of dynamical system theory. In the inertial frame these equilibrium points appear as ***periodic orbits***.

Equations of Motion of the PCR3BP

Geometry of Rotating Frames.

- Let XY be the inertial frame and xy be the rotating frame. Assume that they coincide at $t = 0$. Then (since the angular velocity is unity)

$$X = x\cos t - y\sin t,$$
$$Y = x\sin t + y\cos t.$$

The situation is the same as one encounters in Calculus (see the figure).

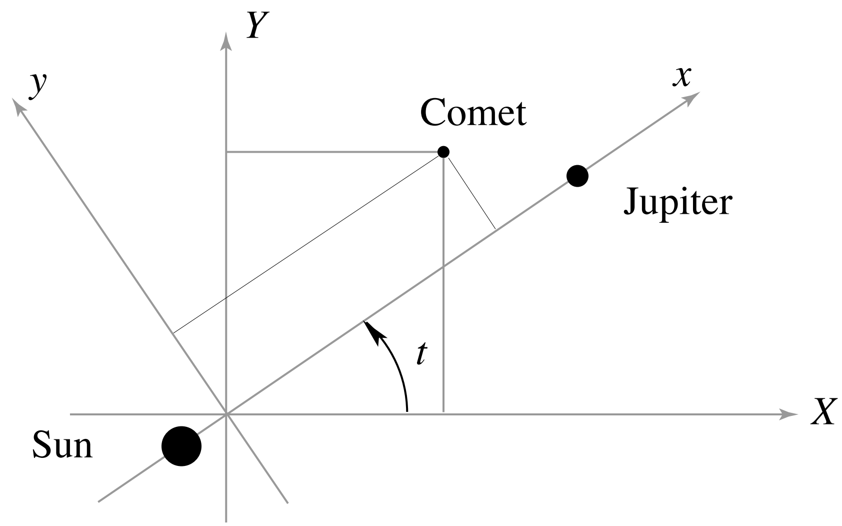


FIGURE 1: The geometry of rotating coordinates.

- In the inertial frame, the Sun is at $(-\mu, 0)$, Jupiter is at $(1-\mu, 0)$ when $t = 0$. At general times,

$$(X_1, Y_1) = (-\mu \cos t, -\mu \sin t),$$
$$(X_2, Y_2) = ((1-\mu) \cos t, (1-\mu) \sin t)$$

are the positions of the Sun and Jupiter respectively in the inertial frame.

- Let (X, Y) be the postion of the comet (or spacecraft, etc) in the inertial frame, then the gravitational potential due to the Sun and Jupiter is (in normalized units)

$$\mathcal{U} = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

where r_1 and r_2 are the distances of the comet from the Sun and Jupiter respectively and are given by

$$\begin{aligned} r_1^2 &= (X + \mu \cos t)^2 + (Y + \mu \sin t)^2, \\ r_2^2 &= (X - (1-\mu) \cos t)^2 + (Y - (1-\mu) \sin t)^2. \end{aligned}$$

Equations of Motion

- **Method 1: Newtonian approach–inertial frame.** In the inertial frame, the Newtonian equations of motion are

$$\ddot{X} = -\mathcal{U}_X, \qquad \ddot{Y} = -\mathcal{U}_Y,$$

where $\mathcal{U}_X$ and $\mathcal{U}_Y$ are the partial derivatives of $\mathcal{U}$ with respect to X, Y respectively. This system is ***non-autonomous***. One can now make a transformation of variables to the variables (x, y) by direct computation (see Marsden and Ratiu [1999] for this type of calculation). This procedure leads to the same equations of motion in terms of (x, y) as the methods below.

- Recall the general form of the ***Euler-Lagrange equations***:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0,$$

where the mechanical system is described by generalized coordinates $(q^1, \ldots, q^n)$. Often one chooses the Lagrangian to be the kinetic minus the potential energies. See Marsden and Ratiu [1999] or other books on mechanics for a discussion.

- **Method 2: Lagrangian approach–inertial frame.** In the inertial frame, the Lagrangian is kinetic minus potential energies:

$$L(X, Y, \dot{X}, \dot{Y}, t) = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) - \mathcal{U}(X, Y, t)$$

The Euler-Lagrange equations are exactly the same as the Newtonian equtions.

- **Method 3: Lagrangian approach—rotating frame.** In the rotating frame, the Lagrangian L_R is given by

$$L_R(x, y, \dot{x}, \dot{y}) = \frac{1}{2}((\dot{x} - y)^2 + (\dot{y} + x)^2) - U(x, y),$$

where the ***gravitational potential*** in rotating coordinates is

$$U = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2}.$$

Reason:

$$\begin{aligned} \dot{X} &= (\dot{x} - y)\cos t - (\dot{y} + x)\sin t, \\ \dot{Y} &= (x + \dot{y})\cos t - (\dot{x} - y)\sin t, \end{aligned}$$

which yields $\dot{X}^2 + \dot{Y}^2 = (\dot{x} - y)^2 + (\dot{y} + x)^2$. Also, since both the distances r_1 and r_2 are invariant under rotation, we have

$$\begin{aligned} r_1^2 &= (x + \mu)^2 + y^2, \\ r_2^2 &= (x - (1 - \mu))^2 + y^2. \end{aligned}$$

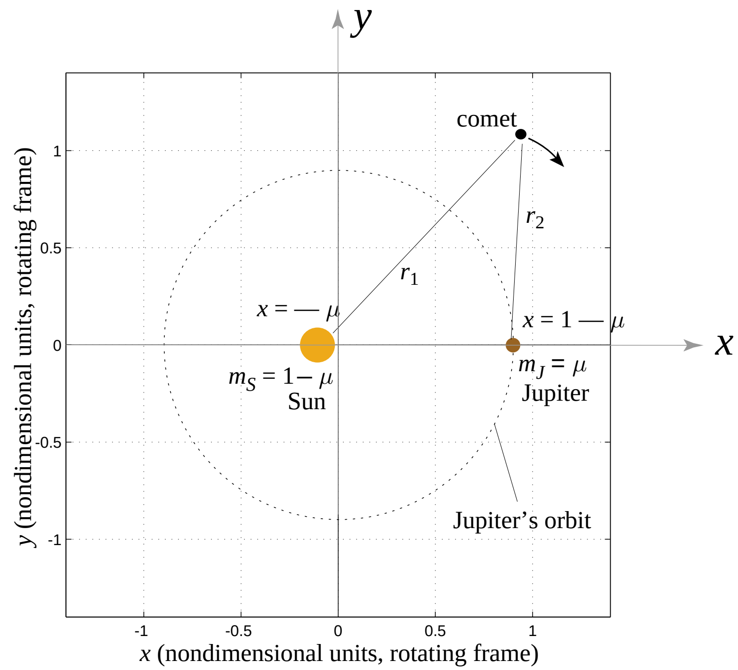


FIGURE 2: The distances r_1 and r_2 in the rotating frame.

- The theory of ***moving systems*** says that one can simply write down the Euler-Lagrange equations in the rotating frame and one will get the correct equations. It is a very efficient method.

- In the present case, the Euler-Lagrange equations are given by

$$\begin{aligned}\frac{d}{dt}(\dot{x} - y) &= \dot{y} + x - U_x, \\ \frac{d}{dt}(\dot{y} + x) &= -\dot{x} + y - U_y.\end{aligned}$$

After simplification, we have

$$\boxed{\ddot{x} - 2\dot{y} = -U_x^{\text{eff}}, \quad \ddot{y} + 2\dot{x} = -U_y^{\text{eff}}}$$

where

$$U^{\text{eff}} = U(x, y) - \frac{1}{2}(x^2 + y^2)$$

is the ***augmented or effective potential*** and the subscripts denote its partial derivatives.

This form of the equations is discussed in detail in Szebeheley [1967] and may be more familiar to the astronomy and astrodynamics communities.

- Recall that whenever one has a Lagrangian system, one can transform it to Hamiltonian form by means of the ***Legendre transformation***:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}; \quad H(q^i, p_i) = \sum_{i=1}^{n} p_i \dot{q}^i - L(q^i, p_i)$$

to get the equations in ***Hamiltonian form***

$$\dot{q}^i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

- **Method 4: Hamiltonian approach–rotating frame**

In our case, the Legendre transformation is given by

$$p_x = \frac{\partial L_R}{\partial \dot{x}} = \dot{x} - y,$$

$$p_y = \frac{\partial L_R}{\partial \dot{y}} = x + \dot{y},$$

we obtain the Hamiltonian function

$$\begin{aligned} H_R(x, y, p_x, p_y) &= p_x \dot{x} + p_y \dot{y} - L_R \\ &= \frac{1}{2}((p_x + y)^2 + (p_y - x)) + U^{\text{eff}}(x, y). \end{aligned}$$

where p_x and p_y are the conjugate momenta.

Hence the ***Hamiltonian equations*** are given by

$$\begin{aligned} \dot{x} &= \frac{\partial H_R}{\partial p_x} = p_x + y, \\ \dot{y} &= \frac{\partial H_R}{\partial p_y} = p_y - x, \\ \dot{p}_x &= -\frac{\partial H_R}{\partial x} = p_y - x - U_x^{\text{eff}}, \\ \dot{p}_y &= -\frac{\partial H_R}{\partial y} = -p_x - y - U_y^{\text{eff}}. \end{aligned}$$

One can also transform from the inertial frame to the rotating frame by using the theory of canonical transformations. This method, while the one classically used, is more complicated. See Whittaker's book for details.

Notice that both the Lagrangian and the Hamiltonian form of the equations in rotating coordinates (x, y) give an autonomous system. Viewed as a dynamical system, it is a four dimensional dynamical system in either $(x, y, \dot{x}, \dot{y})$ or (x, y, p_x, p_y) space.

■ Energy Integral and Jacobi Constant.

Since equations of motion of the PCR3BP are Hamiltonian and autonomous, they have an energy integral of motion.

Notation. We use H when we regard the energy as a function of ***positions and momenta*** and E when we regard it as a function of the ***positions and velocity***.

In the astronomy and astrodynamics communities, it is called the ***Jacobi integral***, which is given by

$$C(x, y, \dot{x}, \dot{y}) = -(\dot{x}^2 + \dot{y}^2) - 2U^{\text{eff}}(x, y) = -2E(x, y, \dot{x}, \dot{y}).$$

Usually in those communities, the existence of the Jacobi integral is derived directly from the equations of motion. The computation is straightforward:

$$\frac{d}{dt}(\dot{x}^2 + \dot{y}^2) = 2(\dot{x}\ddot{x} + \dot{y}\ddot{y}) = 2\frac{d}{dt}(-U^{\text{eff}}),$$

so we get

$$0 = \frac{d}{dt}\left(-2U^{\text{eff}}(x, y) - (\dot{x}^2 + \dot{y}^2)\right) = \frac{d}{dt}C$$

Equilibria of the PCR3BP

- To find equilibria, we set the right hand sides of the system equal to zero.
- Since the system is

$$\begin{aligned}\dot{x} &= v_x,\\ \dot{y} &= v_y,\\ \dot{v}_x &= 2v_y - U_x^{\text{eff}},\\ \dot{v}_y &= -2v_x - U_y^{\text{eff}}.\end{aligned}$$

  we see that equilibia in $(x, y, \dot{x}, \dot{y})$ space are of the form $(x_e, y_e, 0, 0)$, where (x_e, y_e) are ***critical points of the effective potential function*** U^{eff}.
- The system has ***five equilibrium points***, three ***collinear equilibria*** on the x-axis, called L_1, L_2, L_3 and two ***equilateral***

points called L_4, L_5.

- These were discovered by Euler and Lagrange in the 1700s.

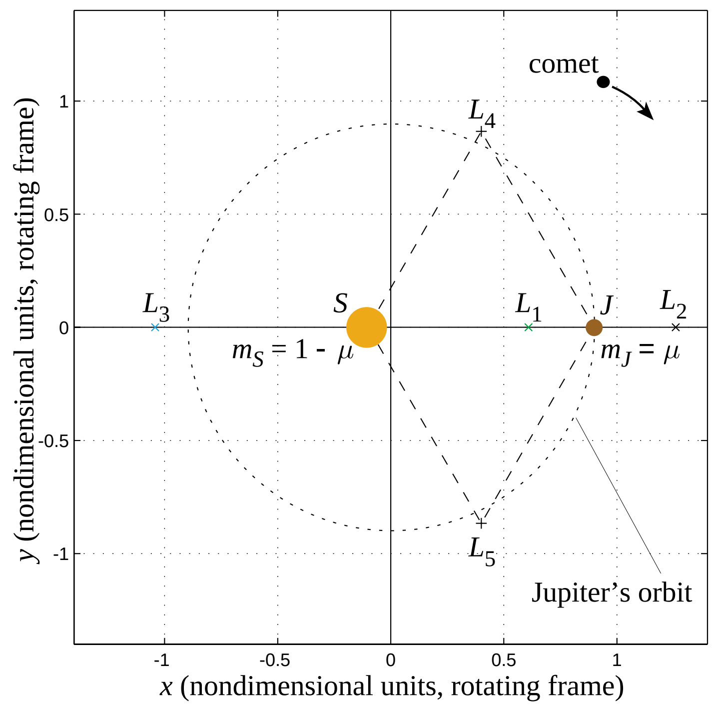


FIGURE 3: The equilibrium points in the PCR3BP.

These equilibria can be found as follows.

The equilateral points

- First, seek solution that do not lie on the line joining the primaries, i.e., $y \neq 0$.
- Using the distances r_1, r_2 as variables and the relation
$$x^2 + y^2 = (1 - \mu) r_1^2 + \mu r_2^2 - \mu(1 - \mu)$$
we see that U^{eff} can be written as
$$-U^{\text{eff}}(r_1, r_2) = \frac{1}{2}(1 - \mu) r_1^2 + \frac{1}{2} \mu r_2^2 + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}.$$
- Using the chain rule, it is straightforward to show that if $y \neq 0$, then $U^{\text{eff}}(r_1, r_2)$ and $U^{\text{eff}}(x, y)$ have the same critical points.

Solving the following systems

$$0 = -U_{r_1}^{\text{eff}} = \mu r_2 - \frac{\mu}{r_2^2},$$
$$0 = -U_{r_2}^{\text{eff}} = (1-\mu) r_1 - \frac{(1-\mu)}{r_1^2},$$

we get the unique solution $r_1 = r_2 = 1$.

- This solution lies at the vertex of an equilateral triangle whose base is the line segment joining the two primaries. By convention, the one in the upper half-plane is denoted L_4, and the one in the lower half-plane is denoted L_5. These are attributed to Lagrange.

The Collinear Points

- Now consider equilibria along the line of primaries where $y = 0$.
- In this case the effective potential function has the form

$$-U^{\text{eff}}(x,0) = \frac{1}{2}x^2 + \frac{1-\mu}{|x+\mu|} + \frac{\mu}{|x-1+\mu|}.$$

- By elementary calculus, it can be determined that $U^{\text{eff}}(x,0)$ has precisely one critical point in each of the following three intervals:
  - $(-\infty, -\mu)$,
  - $(-\mu, 1-\mu)$ and
  - $(1-\mu, \infty)$.
- A sketch of the graph of $U^{\text{eff}}(x,y)$ is given in the figure. These three collinear equilibria are attributed to Euler and are denoted by L_1, L_2, L_3. For more details, see Meyer and Hall [1992].

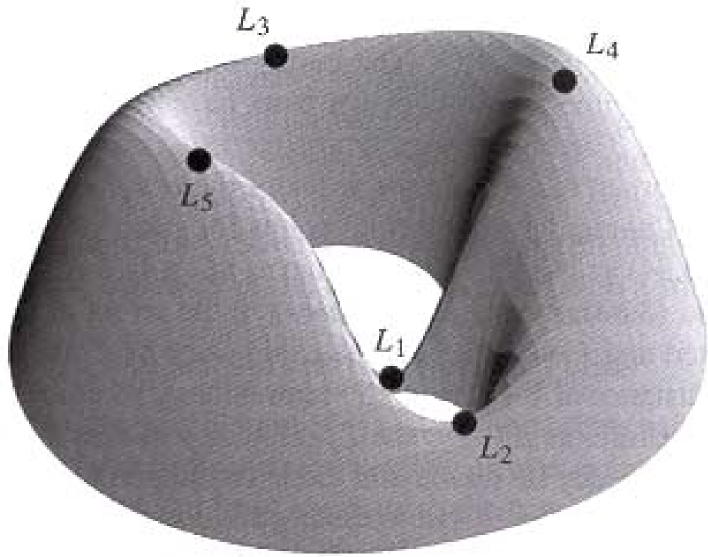


FIGURE 4: Graph of the effective potential for the Sun-Jupiter system.

Locating the Collinear Equilibria.

Computation of the values of the abscissas of the collinear points requires the solution of $\frac{d}{dx}U^{\text{eff}}(x,0) = 0$, which is a quintic equation after simplification.

Historically, a lot of effort has been spent in finding series expansions for such solutions. Here, we will write down two of those that are most useful for us, namely, the distances $d(m_2, L_1), d(m_2, L_2)$ between the smaller primary and the L_1 and L_2 respectively.

$$d(m_2, L_1) = \nu \left(1 - \frac{1}{3}\nu - \frac{1}{9}\nu^2 + \dots \right),$$
$$d(m_2, L_2) = \nu \left(1 + \frac{1}{3}\nu - \frac{1}{9}\nu^2 + \dots \right),$$

where $\nu = (\frac{\mu}{3})^{1/3}$. Of course locating these points numerically is no problem.

■ **Linearization around L_1 and L_2** Linearizing the PCR3BP equations around one of the equilibrium points L_1, or L_2, we get the linear equations

$$\begin{aligned}\dot{x} &= v_x,\\ \dot{y} &= v_y,\\ \dot{v}_x &= 2v_y + ax,\\ \dot{v}_y &= -2v_x - by,\end{aligned}$$

where the values of a and b are specific values for each of the equilibria. For example, they are approixmately $a = 9.892$ and $b = 3.446$ for L_1 in the Sun-Jupiter system.

■ **Instability of L_1 and L_2** It is straightforward to find that the eigenvalues of this linear system have the form $\pm\lambda$ and $\pm i\nu$ where λ and ν are positive constants. Therefore, all the collinear equilibria are unstable and have the characteristic of saddle $\times$ center.

The Energy Manifold and Hill's region

How to obtain Hill's region.

- Recall that the system has an integral

$$E(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U^{\text{eff}}(x, y) = -\frac{C}{2}.$$

- This energy integral will help us determine the region of possible motion, i.e., which region in which the comet can possibly move along and the region in which it is forbidden to move.
- The first step is to look at the surface of the effective potential again and make some remarks about it.

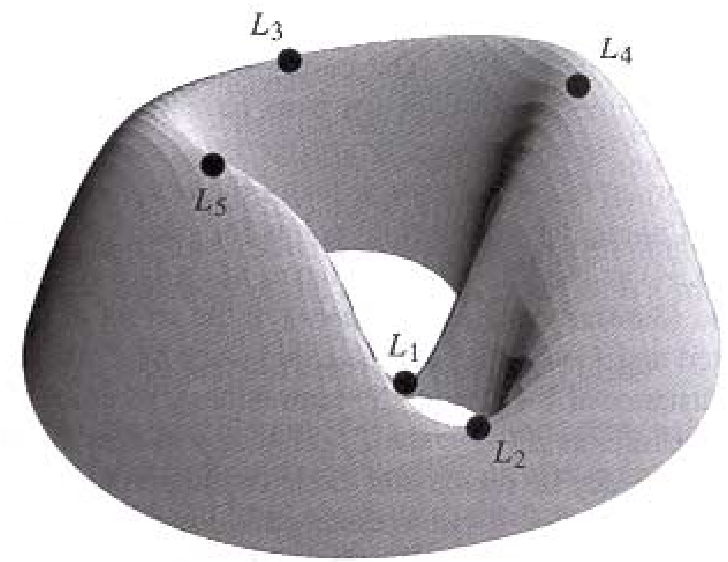


FIGURE 5: Graph of the effective potential for the Sun-Jupiter system.

- Near either the Sun and Jupiter, we have a potential well.
- Far away from the Sun-Jupiter system, the term that corresponds to the centrifugal force dominates, we have another potential well.
- Moreover, by applying multivariable calculus, one finds that

there are 3 saddle points at L_1, L_2, L_3 and 2 maxima at L_4 and L_5.

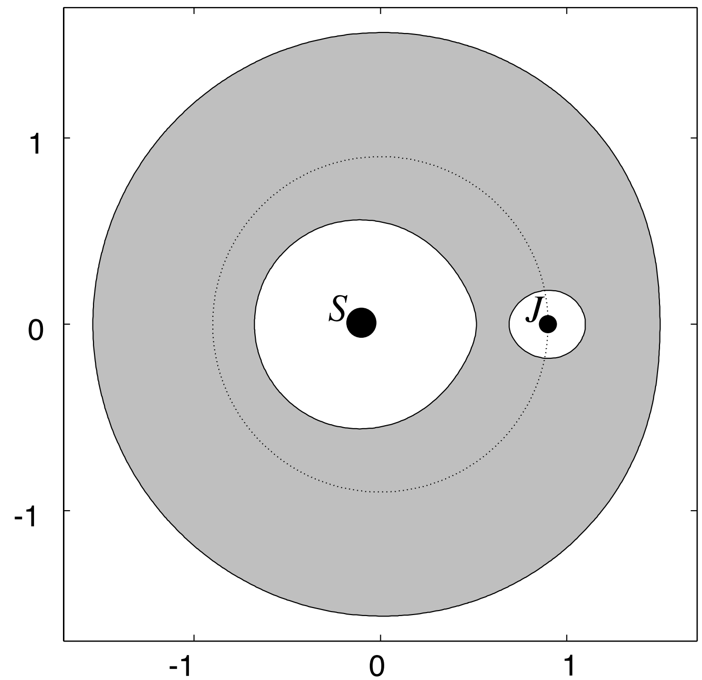
- Let E_i be the energy at L_i; then

$$E_5 = E_4 > E_3 > E_2 > E_1.$$

- **Five cases.** Fixing the energy E is like fixing a height in this plot of the effective potential.

Contour plots give 5 cases of Hill's region. The white area is the ***Hill's region*** and the gray area is the ***forbidden region***.

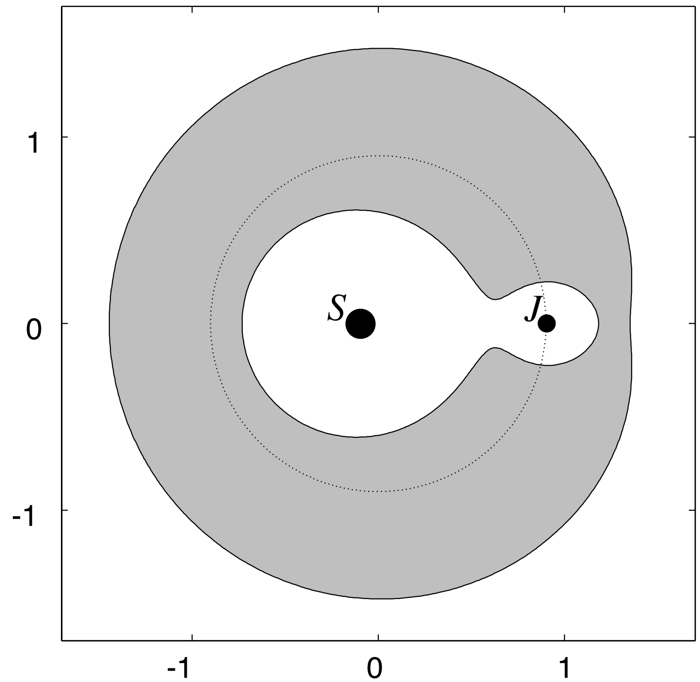
- **Case 1.** If the energy of the comet is below E_1, the comet cannot move between the Sun and Jupiter region.



Case 1. $E < E_1$ or $C > C_1$

FIGURE 6: Hill's region—Case 1.

- **Case 2.** If the energy is just above E_1, the comet can now move between the Sun and Jupiter region. But it is still barred from moving between the Sun and Exterior region.



Case 2. $E_1 < E < E_2$ or $C_1 > C > C_2$

FIGURE 7: Hill's region—Case 2.

- **Case 3.** The case that concerns us the most is when the energy is just above E_2. Now the comet can move between the Sun and Exterior region passing through Jupiter region.

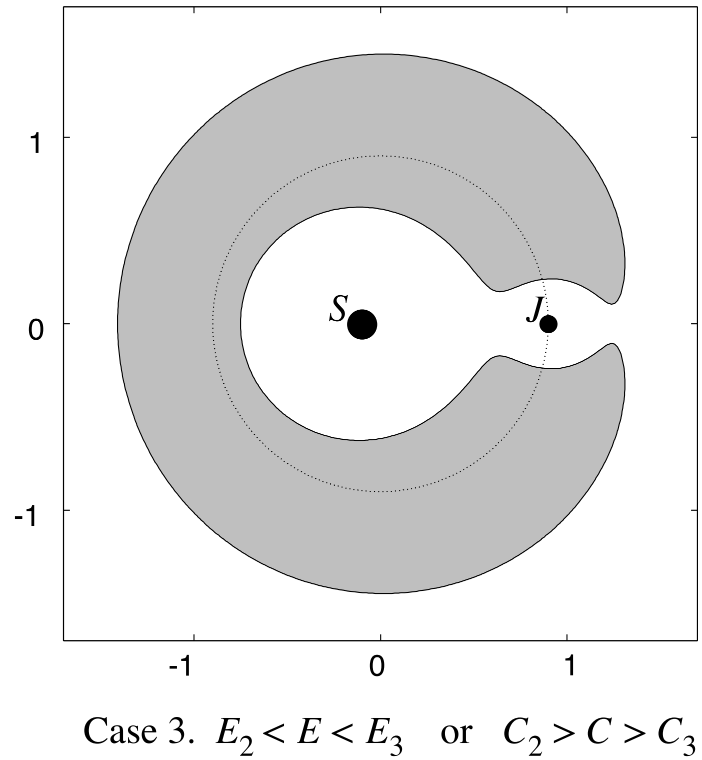
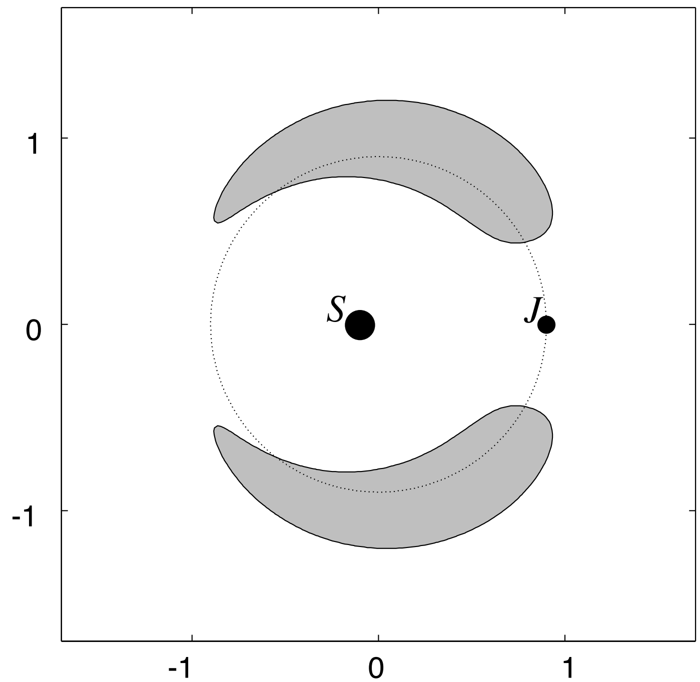


FIGURE 8: Hill's region—Case 3.

- **Case 4.** In this case the energy is energy is above E_3 but below that of E_4 and E_5.



Case 4. $E_3 < E < E_4 = E_5$ or $C_3 > C > C_4 = C_5$

FIGURE 9: Hill's region—Case 4.

Case 5: If $E > E_4 = E_5$, the forbidden region dissappears.

The 3-Dimensional CR3BP Problem

- Of course it is important that one extends the planar problem to the three dimensional case as well. The planar problem sits inside this larger model as an ***invariant subsystem***.
- Using nondimensional units, the equations of motion are

$$
\begin{aligned}
\dot{x} &= u \\
\dot{y} &= v \\
\dot{z} &= w \\
\dot{u} &= 2y - U_x^{\text{eff}} \\
\dot{v} &= -2x - U_y^{\text{eff}} \\
\dot{w} &= -U_z^{\text{eff}},
\end{aligned}
$$

where

$$-U^{\text{eff}} = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{d_1} + \frac{\mu}{d_2}$$

$$d_1 = \left((x+\mu)^2 + y^2 + z^2\right)^{1/2}$$

$$d_2 = \left((x-1+\mu)^2 + y^2 + z^2\right)^{1/2}$$

and where μ is defined as before. As before, time is scaled by the period of the primaries orbits ($T/2\pi$, where $T = 1$ year), positions are scaled by the Sun-Earth distance ($L = 1.49597927 \cdot 10^8$km), and velocities are scaled by the Earth's average orbital speed around the Sun ($2\pi L/T = 29.80567$km/s).